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The Riemann Integral

Partition of a closed Interval

Let $[a.b]$ be a finite closed interval
If $a = x_0 \le x_1 \le x_2 \le \cdots \le x_n = b$
, then finite ordered set $P = \{a = x_0, x_1, \ldots x_n\}$
is called a partition of $[a.b]$

$n+1$ points $x_0, x_1, \ldots x_n$ are called partition points of P. These n closed sub-intervals $[x_0 . x_1], [x_1, x_2] \ldots [x_{n-1}, x_n]$ are called segments of the partition P

Notation we use Δx_i to denote the ith interval and also its length.

Thus $\Delta x_i = x_i - x_{i-1}$

$$\sum_{i=1}^{n} \Delta x_i = \Delta x_1 + \Delta x_2 + \Delta x_3 + \cdots + \Delta x_n$$
$$= b - a.$$

Norm of a Partition

The maximum of the lengths of the sub-intervals of a partition P is called the norm or mesh of partition P and is denoted by $\|P\|$ or $\mu(P)$. OR The length of the largest of the sub-intervals of a partition P is called norm of P

Thus $\|P\| = Max\{\Delta x_i : i=1,2 \cdots n\}$

$= Max\{x_i - x_{i-1} : i=1,2 \cdots n\}$

$= \max_{i=1}^{n} \Delta x_i$

Refiniment of Partition

If P_1, P_2 are two partitions of $[a,b]$ and $P_1 \subseteq P_2$, then P_2 is called a refiniement of partition P_1

Thus if P_2 is finer than P_1, then every point of P_1 is used in P_2 and P_2 has at least one additional point

Common Refinement

If P_1, P_2 are two partitions of $[a,b]$, then $P_1 \cup P_2$ is called common refinent of P_1 & P_2 because

$$P_1 \subseteq P_1 \cup P_2 \ \& \ P_2 \subseteq P_1 \cup P_2$$

Note (1) Partition is also known as dissection or net

(2) By changing the partition points the partition can be changed and hence there can be an infinite no of partitions of the interval $[a,b]$

(3) The set of all partitions of $[a,b]$ will be denoted by $P[a,b]$

(3) Upper And Lower Darboux Sums

Let $f: [a \cdot b] \longrightarrow R$ be a bounded functio

i.e f is real valued bound function defined on $[a.b]$

and $P = \{a = x_0, x_1, x_2 \dots x_n\}$ be a partiti

of $[a\ b]$.

$\because$ f is bounded on $[a\ b]$

$\therefore$ f is bonded on each sub-interval

Let $M = \operatorname{Sup} f$ on $[a\ b]$

$m = \operatorname{Inf} f$ on $[a\ b]$

$M_i = \operatorname{Sup} f$ on $[x_{i-1}, x_i]$

$m_i = \operatorname{Inf} f$ on $[x_{i-1}, x_i]$

Then upper and lower Darboux sums of f Corresponding to partition P are denoted by $U(P,f)$ or $U(f,P)$ & $L(P.f)$ or $L(f.P)$ and are defined as

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

Remarks# Clearly the sums $U(P,f)$ & $L(P,f)$ depend upon function f and partition P and do exist for every bounded function (f)

☆ By Muhammad Hussain Lecturer
Govt College Asghar Mall RWP

Oscillatory sum & Oscillation

Let f be real valued function defined and bounded on $[a, b]$ and $P = \{x_0, x_1, x_2 \dots x_n\}$ be a partition of $[a, b]$.

Let $M_i = \sup f$ on $[x_{i-1}, x_i]$

$m_i = \inf f$ on $[x_{i-1}, x_i]$

Then

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i$$

$$= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$$

$$= \sum_{i=1}^{n} O_i \Delta x_i$$

where $O_i = M_i - m_i$ denotes the oscillation of f in $[x_{i-1}, x_i]$ and $\sum_{i=1}^{n} O_i \Delta x_i$ is called oscillatory sum of f corresponding to partition P and is denoted by $\omega(P, f)$

$\because O_i = M_i - m_i \geq 0$

$\therefore \omega(P, f) \geq 0$

Explaination of Upper & Lower Darboux Sums

Consider a simple partition $P = \{x_0 = a, x_1, x_2, x_3, x_4, x_5 = b\}$ and suppose that f has a simple graph as shown

(a) $L(P, f)$ = total area in fig ① i.e. it is the total area of inscribed Rectangles of curve $y = f(x)$ from $x = a$ to $x = b$

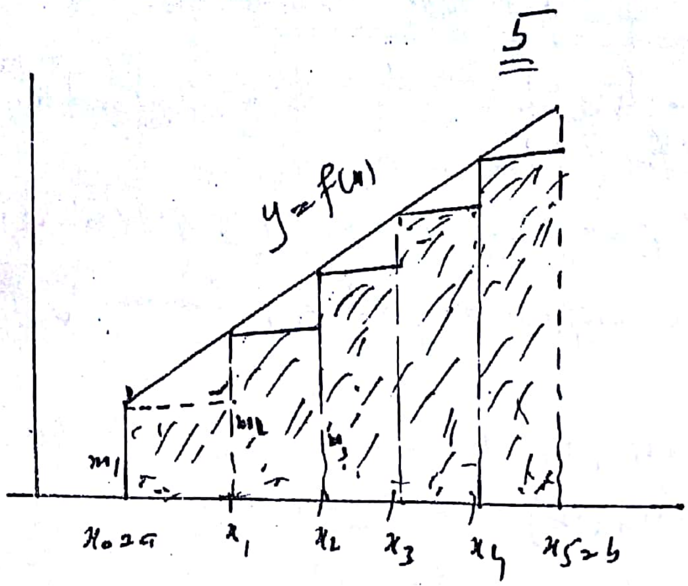
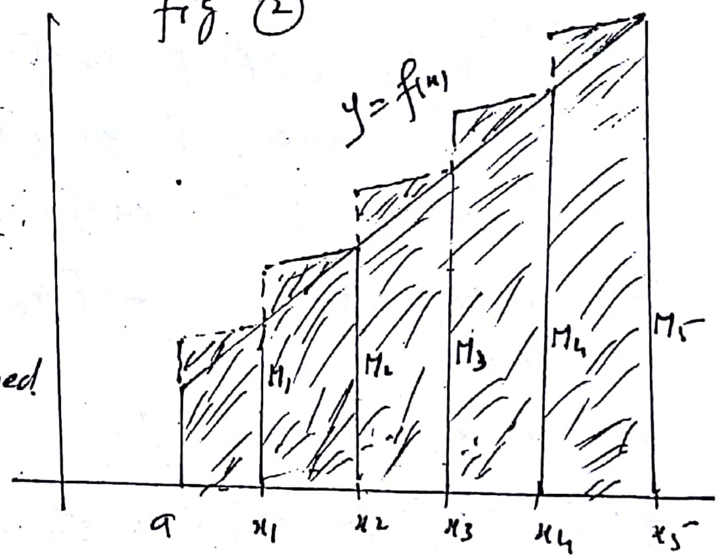


fig ①

It approximate area under the curve from below.

fig ②



(b) $U(P, f)$ = total shaded area in fig ② i.e. it is the total area of circumscribed rectangles of the curve $y = f(x)$ from $x = a$ to $x = b$ and it approximate the area under f from above.

(c) If A is the exact area under the graph of f from $x = a$ to $x = b$, then

$$L(P, f) \leq A \leq U(P, f)$$

<u>Note</u> # when norm (mesh) of partition P is decreased by increasing the no of points of division (intermediate) points, lower sum increases and upper sum decreases. Also

$$L(P, f) \leq U(P, f)$$

Properties of Darboux Sums # 6

<u>Theorem</u> 1 # If $f : [a,b] \to R$ is bounded & P is any partition of, Then

$$m(b-a) \le L(P.f) \le U(P.f) \le M(b-a)$$

Where m, M are infinum and supremum of f on $[a.b]$

<u>Proof</u> let $P = \{a = x_0, x_1, x_2 \cdots x_n = b\}$ be a partition of $[a.b]$

$\because$ f is bounded on $[a, b]$

$\therefore$ f is bounded on each sub-interval $[x_{i-1}, x_i]$

Let $m_i = \text{Inf } f$ on $[x_{i-1}, x_i]$

$M_i = \text{Sup } f$ on $[x_{i-1}, x_i]$

$M = \text{Sup } f$ on $[a, b]$

$m = \text{Inf } f$ on $[a \; b]$

Then

$$m \le m_i \le M_i \le M$$

$$\Rightarrow m \Delta x_i \le m_i \Delta x_i \le M_i \Delta x_i \le M \Delta x_i$$

$$\Rightarrow m \sum_{i=1}^{n} \Delta x_i \le \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i \le M \sum \Delta x_i$$

$$m(b-a) \le L(P.f) \le U(P.f) \le M(b-a) \to (1)$$

$\Rightarrow$ $L(P,f)$ & $U(P.f)$ are bounded if f is bounded.

<u>Note</u> # By Considering all partitions of $[a, b]$ we get a set U of upper sums and set L of lower sums. The inequality (1) shows That both these are bounded. and each set has supremum and infimum

(2) <u>Theorem</u># If $f : [a, b] \longrightarrow R$ is a bounded function and p is any partition of $[a, b]$

(i) $L(P, f) \leq U(P, f)$ (ii) $L(P, -f) = -U(P, f)$

and

$U(P, -f) = -L(P, f)$

<u>Note</u> If $A \subseteq R$, then let $-A = \{x \in R \mid -x \in A\}$

A is bounded $\Leftrightarrow$ $-A$ is bounded.

and $\sup(-A) = -\inf A$ & $\inf(-A) = -\sup A$

Now if $f \in \beta[a, b]$, then $-f \in \beta[a, b]$

<u>Proof</u> Let $P = \{a = x_0, x_1, x_2 - - - x_n = b\}$ be any partition of $[a, b]$. Let m_i, M_i inf & sup f on $[x_{i-1}, x_i]$ & M, m be sup & inf of f on $[a, b]$. Then.

$$m_i \leq M_i \quad i = 1, 2, 3 - - n$$

$$\Rightarrow m_i \Delta x_i \leq M_i \Delta x_i \quad i = 1, 2, 3 - - n$$

$$\Rightarrow \sum m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i$$

$$\Rightarrow L(P, f) \leq U(P, f)$$

$\because$ f is bounded on $[a, b]$

$\therefore$ $-f$ is bounded on $[a, b]$

$$\sup(-f(x)) = -\inf f(x) \qquad x \in [x_{i-1}, x_i]$$

$$= -m_i$$

$$\inf(-f(x)) = -\sup f(x) \qquad x \in [x_{i-1}, x_i]$$

$$= -M_i$$

Hence

$$U(P, -f) = \sum_{i=1}^{n} -m_i \Delta x_i = -L(P, f)$$

$$L(P, -f) = \sum_{i=1}^{n} -M_i \Delta x_i = -\sum_{i=1}^{n} M_i \Delta x_i$$
$$= -U(P, f)$$

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3) **Theorem #** If P_1, P_2 are any two partitions of $[a, b]$ and $P_1 \subseteq P_2$, Then

(i) $L(P_2, f) \geq L(P_1, f)$ lower sum cannot decrease with refineness of partition

(ii) $U(P_2, f) \leq U(P_1, f)$ The upper sum cannot increase by the refinement of partition

(iii) $L(P_1, f) \leq L(P_2, f) \leq U(P_2, f) \leq U(P_1, f)$

(iv) $U(P_1, f) - L(P_1, f) \geq U(P_2, f) - L(P_2, f)$

(v) $\omega(P_1, f) \geq \omega(P_2, f)$

Proof (i) Suppose that $P_1 = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ and P_2 be a partition with k additional points than those of P_1 such that

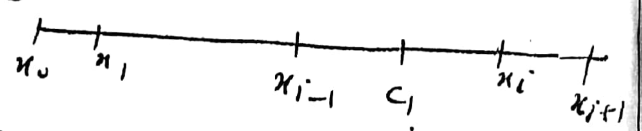
$$x_{i-1} < c_i < x_i \quad \forall\, i = 1, 2, \ldots n$$

Let $P_2' = \{c_1\} \cup P_1$ where $x_{i-1} < c_1 < x_i$

let $m_i = \text{Inf } f$ on $[x_{i-1}, x_i]$

$w_1 = \text{Inf } f$ on $[x_{i-1}, c_1]$

$w_2 = \text{Inf } f$ on $[c_1, x_i]$



Then clearly

$m_i \leq w_1$ & $m_i \leq w_2$

$$L(P_2', f) - L(P_1, f) = \sum_{i=1}^{i-1} m_i \Delta x_i + w_1(c_1 - x_{i-1}) + w_2(x_i - c_1) + \sum_{i=i+1}^{n} m_i \Delta x_i \ldots$$

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$$= \sum_{i=1}^{i-1} \cancel{m_i \Delta x_i} + \omega_1 (c_1 - x_{i-1}) + \omega_2 (x_i - c_1)$$

$$+ \sum_{i=i+1}^{n} \cancel{m_i \Delta x_i} - \sum_{i=1}^{i-1} \cancel{m_i \Delta x_i} - m_i (x_i - x_{i-1})$$

$$- \sum_{i=i+1}^{n} \cancel{m_i \Delta x_i}$$

$$= \omega_1 (c_1 - x_{i-1}) + \omega_2 (x_i - c_1) - m_i (x_i - x_{i-1})$$

$$= \omega_1 (c_1 - x_{i-1}) + \omega_2 (x_i - c_1) - m_i [x_i - c_1 + c_1 - x_{i-1}]$$

$$\Rightarrow (\omega_1 - m_i)(c_1 - x_{i-1}) + (\omega_2 - m_i)(x_i - c_1)$$

$\because$ $\omega_1 - m_i \geq 0$ $\quad c_1 - x_{i-1} \geq 0$ $\quad \omega_2 - m_i \geq 0$

$x_i - c_1 \geq 0$

$$\Rightarrow L(P_2', f) - L(P_1, f) \geq 0$$

$$\Rightarrow L(P_2', f) \geq L(P_1, f) \longrightarrow (1)$$

let $P_2'' = \{c_2\} \cup P'$, then by above argument

$$L(P'', f) \geq L(P_1', f) \longrightarrow (2)$$

Let $P_2''' = \{c_3\} \cup P''$

$$L(P_2''', f) \geq L(P_2'', f) \longrightarrow (3)$$

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let $P_2^k = P_2^{k-1} \cup \{c_k\} = P_2$

$$L(P_2^k, f) \geq L(P_2^{k-1}, f) \longrightarrow (k)$$

Adding all these inequatities we get

$$L(P_2^k, f) \geq L(P_1 f)$$

$$\Rightarrow L(P_2 f) \geq L(P_1, f)$$

(ii) $U(P_1, f) - U(P_2', f)$

$$= \sum_{i=1}^{n} M_i \Delta x_i - \left[\sum_{i=1}^{i-1} M_i \Delta x_i + W_1 (c_1 - x_{i-1}) + W_2 (x_{i-1} - c_1) + \sum_{i=i+1}^{n} M_i \Delta x_i \right]$$

where $W_1 = \sup f$ on $[x_{i-1}, c_1]$
$W_2 = \sup f$ on $[c_1 \quad x_i]$

$$= \cancel{\sum_{i=1}^{i-1} M_i \Delta x_i} + M_i (x_i - x_{i-1}) + \cancel{\sum_{i=i+1}^{n} M_i \Delta x_i} - \cancel{\sum_{i=1}^{i-1} M_i \Delta x_i} - W_1 (c_1 - x_{i-1}) - W_2 (x_i - c_1) - \cancel{\sum_{i=i+1}^{n} M_i \Delta x_i}$$

$$= M_i (x_i - x_{i-1}) - W_1 (c_1 - x_{i-1}) - W_2 (x_{i-1} - c_1)$$

$$= M_i [x_i - c_1 + c_1 - x_{i-1}] - W_1 (c_1 - x_{i-1}) - W_2 (x_i - c_1)$$

$$= (M_i - W_2)(x_i - c_1) + (M_i - W_1)(c_1 - x_{i-1}) \geqslant 0$$

$\because$ $M_i - W_2 \geqslant 0 \quad M_i - W_1 \geqslant 0 \quad x_i - c_1 \geqslant 0$
$c_1 - x_{i-1} \geqslant 0$

$\Rightarrow \quad U(P_1, f) \geqslant U(P_2', f) \longrightarrow (1)$

let $P_2'' = P_2' \cup \{c_2\}$, Then
$U(P_2', f) \geqslant U(P_2'', f) \longrightarrow (2)$

Let $P_2''' = P_2'' \cup \{c_3\}$, Then
$U(P_2'', f) \geqslant U(P_2''', f) \longrightarrow (3)$

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let $P_2^{k} = P_2^{k-1} \cup \{c_k\} = P_2$, Then
$U(P_2^{k-1}, f) \geqslant U(P_2^{k}, f) \longrightarrow (k)$

Adding all these inequalities, we get

$$U(P_1,f) \geq U(P_2^k,f) = U(P_2,f)$$

$$\Rightarrow U(P_1,f) \geq U(P_2,f)$$

(iii) Now $U(P_1,f) \geq U(P_2,f)$

$$\boxed{\begin{array}{l} \cancel{L(P_1,f) \leq L(P_2,f)} \\ \therefore \cancel{-L(P_1,f) \geq -L(P_2,f)} \end{array}}$$

$$L(P_1 f) \leq L(P_2,f)$$

$$L(P_2,f) \leq U(P_2,f)$$

Combining these

$$L(P_1 f) \leq L(P_2,f) \leq U(P_2,f) \leq U(P_1,f)$$

(iv) $$U(P_1,f) \geq U(P_2,f) \rightarrow ①$$

$$L(P_1,f) \leq L(P_2,f)$$

$$-L(P_1,f) \geq -L(P_2,f) \rightarrow ②$$

Adding ① & ②

$$U(P_1,f) - L(P_1,f) \geq U(P_2,f) - L(P_1,f)$$

OR

$$L(P_1,f) \leq L(P_2,f) \leq U(P_2,f) \leq U(P_1,f)$$

$$\Rightarrow U(P_1,f) - L(P_1,f) \geq U(P_2,f) - L(P_2,f)$$

[i.e on the no line Distance between $U(P_2 f)$ & $L(P_2,f)$ cannot exceed the distance between $U(P_1,f)$, & $L(P_1,f)$

(iv) It is another form of (iv)

Corollary: If P_2 is a refinement of P_1 containing p points more than P_1 and $|f(x)| \leq k \quad \forall x \in [a, b]$, then

i) $L(P_1, f) \leq L(P_2, f) \leq L(P_1, f) + 2pk\delta$

ii) $U(P_1, f) \geq U(P_2, f) \geq U(P_1, f) - 2pk\delta$

(iii) $\omega(P_1, f) - \omega(P_2, f) \leq 4pk\delta$

where $\|P_1\| = \delta$

Proof: Let P_2' consists of just one point c_1 more than $P_1 = \{a = x_0, x_1, x_2 \dots x_n = b\}$ and $x_{r-1} < c_1 < x_r$, then

$P_2' = \{a = x_0, x_1, x_2 \dots x_{r-1}, c, x_r \dots x_n = b\}$

Let m_r', m_r'' & m_r be infimum of f in the intervals $[x_{r-1}, c_1], [c_1, x_r]$ & $[x_{r-1}, x_r]$ respectively & M_r', M_r'', M_r be supremum of f on these intervals respectively

$\because \quad |f(x)| \leq k \quad \forall x \in [a, b]$

i.e $-k \leq f(x) \leq k \quad \forall x \in [a, b]$

$\Rightarrow -k \leq m_r \leq m_r' \leq k$

$-k \leq m_r \leq m_r'' \leq k$

$\Rightarrow 0 \leq m_r' - m_r \leq 2k.$

& $0 \leq m_r'' - m_r \leq 2k.$

Also $-k \leq M_r' \leq M_r \leq k.$

$-k \leq M_r'' \leq M_r \leq k.$

$\Rightarrow 0 \leq M_r - M_r'' \leq 2k.$

(i) $L(P_2', f) - L(P_1, f) = m_r'(c - x_{r-1}) + m_r''(x_r - c)$
$- m_r(x_r - x_{r-1})$

$= m_k'(c_1 - x_{k-1}) + m_k''(x_k - c_1) - m_k[x_k - c_1 + c_1 - x_{k-1}]$

$= (m_k' - m_k)(c_1 - x_{k-1}) + (m_k'' - m_k)(x_k - c_1)$

$\le 2k(c_1 - x_{k-1}) + 2k(x_k - c_1) = 2k(x_k - x_{k-1})$

$\le 2k\delta \qquad \because \; x_k - x_{k-1} \le \|P_1\| = \delta$

$\Rightarrow L(P_2', f) \le L(P_1, f) + 2k\delta \longrightarrow (1)$

If P_2 contains p points more than P_1, then introducing the addition points one by one and proceeding as above we have.

$P_2'' = P_2' \cup \{c_2\}$, we get

$L(P_2'', f) \le L(P_2', f) + 2k\delta \longrightarrow (2)$

$P_2''' = P_2'' \cup \{c_3\}$

& $L(P_2''', f) \le L(P_2'', f) + 2k\delta \longrightarrow (3)$

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$P_2 = P_2^p = P_2^{p-1} \cup \{c_p\}$ &

$L(P_2^p, f) \le L(P_2^{p-1}, f) + 2k\delta \longrightarrow (p)$

Adding all these inequalities, we get

$L(P_2^p, f) \le L(P_1, f) + 2kp\delta$

$\Rightarrow L(P_2, f) \le L(P_1, f) + 2kp\delta$

(ii) $U(P_1, f) - U(P_2', f)$

$= M_k(x_k - x_{k-1}) - [M_k'(c_1 - x_{k-1}) + M_k''(x_k - c_1)]$

$= M_k[(x_k - c_1) + (c_1 - x_{k-1})] - M_k'(c_1 - x_{k-1}) - M_k''(x_k - c_1)$

$= (M_k - M_k'')(x_k - c_1) + (M_k - M_k')(c_1 - x_{k-1})$

$\le 2k(x_k - c_1) + 2k(c_1 - x_{k-1})$

$\leq 2k(x_r - x_{r-1})$

$\leq 2k\delta$ where $x_r - x_{r-1} \leq \|P_1\| < \delta$

$\Rightarrow U(P_2', f) \geq U(P_1, f) - 2k\delta \rightarrow (1)$

Similarly

$U(P_2'', f) \geq U(P_2', f) - 2k\delta \rightarrow (2)$

$U(P_2''', f) \geq U(P_2'', f) - 2k\delta \rightarrow (3)$

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$U(P_2^{p-1}, f) \geq U(P_2^{p-2}, f) - 2k\delta \rightarrow (P-1)$

$U(P_2^{p}, f) \geq U(P_2^{p-1}, f) - 2k\delta \rightarrow (P)$

Adding all these inequalities we get.

$U(P_2^{p}, f) \geq U(P_1, f) - 2kp\delta$

$U(P_2, f) \geq U(P_1, f) - 2kp\delta.$

Also

$U(P_1, f) \geq U(P_2, f)$

$\Rightarrow U(P_1, f) \geq U(P_2, f) \geq U(P_1, f) - 2kp\delta.$

(iii) $U(P_1, f) - U(P_2, f) \leq 2pk\delta.$

& $L(P_2, f) - L(P_1, f) \leq 2pk\delta.$

Adding these we get

$[U(P_1, f) - L(P_1, f)] - [U(P_2, f) - L(P_2, f)]$

$\leq 4pk\delta$

$\Rightarrow \omega(P, f) - \omega(P_2, f) \leq 4pk\delta.$

5) # **Theorem** For any two partitions P_1, P_2 of $[a, b]$

(i) $L(P_1, f) \leq U(P_2, f)$

(ii) $L(P_2, f) \leq U(P_1, f)$

i.e no lower sum can exceed any upper sum or no upper sum can ever be less than any lower sum.

Proof # Let $P = P_1 \cup P_2$

Then $L(P_1, f) \leq L(P, f)$ & $U(P, f) \leq U(P_2, f)$

Also $L(P, f) \leq U(P, f)$

Combining, we have.

$$L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f)$$

$$\Rightarrow L(P_1, f) \leq U(P_2, f)$$

Also $L(P_2, f) \leq L(P, f)$ & $U(P, f) \leq U(P_1, f)$

& $L(P, f) \leq U(P, f)$

Combining we have.

$$L(P_2, f) \leq L(P, f) \leq U(P, f) \leq U(P_1, f)$$

$$\Rightarrow L(P_2, f) \leq U(P_1, f)$$

6) **Theorem #** If $f : [a, b] \to R$, $g : [a, b] \to R$ are bounded functions and P is any partition of $[a, b]$, then

(i) $U(P, f+g) \leq U(P, f) + U(P, g)$

(ii) $L(P, f+g) \geq L(P, f) + L(P, g)$

(iii) $\omega(P, f+g) \leq \omega(P, f) + \omega(P, g)$

Proof # Let $P = \{a = x_0, x_1, x_2 \cdots x_n = b\}$ be any partition of $[a, b]$

$\because$ f, g are bounded on $[a, b]$

$\therefore$ $f+g$ is bounded on $[a, b]$

Let $M_r = \text{Sup}(f+g)$ on $[x_{r-1}, x_r]$

$m_r = \text{Inf}(f+g)$ on $[x_{r-1}, x_r]$

$M_r' = \text{Sup } f$ on " "

$m_r' = \text{Inf } f$ on " "

$M_r'' = \text{Sup } g$ on " "

$m_r'' = \text{Inf } g$ on " "

$\because$ M_r', M_r'' are supremum of f, g on $[x_{r-1}, x_r]$

$\Rightarrow$ $f(x) \leq M_r'$ & $g(x) \leq M_r''$ $\forall x \in [x_{r-1}, x_r]$

$\Rightarrow$ $f(x) + g(x) \leq M_r' + M_r''$ $\forall x \in [x_{r-1}, x_r]$

$\Rightarrow$ $(f+g)(x) \leq M_r' + M_r''$ $\forall x \in [x_{r-1}, x_r]$

$\Rightarrow$ $M_r' + M_r''$ is an upper bound of $f+g$ on $[x_{r-1}, x_r]$.

But M_r is the least upper bound of $(f+g)$ on $[x_{r-1}, x_r]$

$\Rightarrow$ $M_r \leq M_r' + M_r''$ on $[x_{r-1}, x_r]$ $\forall r$

$\Rightarrow$ $M_r \Delta x_r \leq M_r' \Delta x_r + M_r'' \Delta x_r$ $\forall r$.

$\Rightarrow$ $\sum_{r=1}^{n} M_r \Delta x_r \leq \sum_{r=1}^{n} M_r' \Delta x_r + \sum_{r=1}^{n} M_r'' \Delta x_r$

$\Rightarrow$ $U(P, f+g) \leq U(P, f) + U(P, g)$

(ii) $\because$ m_r', m_r'' are ~~the least~~ ~~lo~~ the infimum of $f. g$ on $[x_{r-1}, x_r]$

$\therefore$ $f(x) \geq m_r'$ $\forall x \in [x_{r-1}, x_r]$ $\forall r$

$g(x) \geq m_r''$ " " "

$f(x) + g(x) \geq m_k' + m_k'' \quad \forall x \in [x_{k-1}, x_k]$

$\Rightarrow (f+g)(x) \geq m_k' + m_k''$ " " "

$\Rightarrow m_k' + m_k''$ is a lower bound of $f+g$ on $I_k = [x_{k-1}, x_k]$.

But m_k is the greatest lower bound of $f+g$ on $I_k = [x_{k-1}, x_k]$

$\Rightarrow \quad m_k \geq m_k' + m_k''$

$\Rightarrow \quad m_k \Delta x_k \geq m_k' \Delta x_k + m_k'' \Delta x_k$.

$\Rightarrow \quad \sum_{k=1}^{n} m_k \Delta x_k \geq \sum_{k=1}^{n} m_k' \Delta x_k + \sum_{k=1}^{n} m_k'' \Delta x_k$

$\Rightarrow \quad L(P, f+g) \geq L(P, f) + L(P, g)$

(iii) $\omega(P, f+g) = U(P, f+g) - L(P, f+g)$

$\leq [U(P,f) + U(P,g)] - [L(P,f) + L(P,g)]$

$\leq [U(P,f) - L(P,f)] + [U(P,g) - L(P,g)]$

$\omega(P, f+g) \leq \omega(P,f) + \omega(P,g)$

7) <u>Theorem</u> #1 Let f, g be defined and bounded on $[a,b]$ and P be any partition of $[a,b]$

Then

(i) $U(cf, P) = c\,U(P,f)$ for any real constant $c \geq 0$

(ii) $U(P, cf) = c\,L(P,f)$ if $c < 0$

(iii) $U(P,f) \leq U(P, |f|)$

<u>Proof</u> (i) Let $P = \{a = x_0, x_1, x_2 \cdots x_n = b\}$ be a partition of $[a,b]$, M_i, m_i be sup & inf. of f on $[x_{i-1}, x_i]$

Then $\sup(cf) = c \sup f = c M_i$ on $[x_{i-1}, x_i]$

$\Rightarrow M_i' = \sup(cf) = c M_i$

$$U(P, cf) = \sum_{i=1}^{n} \cancel{cf}\, c M_i \Delta x_i$$

$$= \sum_{i=1}^{n} c M_i \Delta x_i = c \sum_{i=1}^{n} M_i \Delta x_i$$

$$= c\, U(P, f)$$

(ii) Let $c = -c_1$ where $c_1 > 0$

$$M_i' = \sup(cf) = \sup(-c_1 f)$$

$$= -\operatorname{Inf}(c_1 f)$$

$$= -c_1 m_i \quad \text{where } m_i = \operatorname{Inf} f \text{ on } [x_{i-1}, x_i]$$

$$= c m_i$$

$$U(P, cf) = \sum_{i=1}^{n} M_i' \Delta x_i$$

$$= \sum_{i=1}^{n} c m_i \Delta x_i = c \sum_{i=1}^{n} m_i \Delta x_i$$

$$= c\, L(P, f)$$

(iii) $\because f(x) \le |f(x)| \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow \sup f(x) \le \sup |f(x)| \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow M_i \le M_i' \Rightarrow \sum M_i \Delta x_i \le \sum M_i' \Delta x_i$

$\Rightarrow U(P, f) \le U(P, |f|)$

Similarly $L(P, f) \le L(P, |f|)$

Upper & Lower Riemann Integral

Let $f:[a\ b]\rightarrow R$ be bounded function. Then fore every partition P of $[a,b]$

$$m(b-a)\leq L(P,f)\leq U(P,f)\leq M(b-a)$$

where m, M are the infimum and supremum of f on $[a,b]$

$\Rightarrow$ The set $\{U(P,f): P\in P[a,b]\}$ of upper sums is bounded below by $m(b-a)$ and Therefore has a greatest lower bound.

The set $\{L(P,f): P\in P[a,b]\}$ of lower sums is bounded above by $M(b-a)$ and Therefore has the least upper bound.

Lower Riemann integral is defined as $\sup\{L(P,f): P\in P[a,b]\}$ or $\sup_P L(P,f)$ and is denoted by $\int_a^b f\,dx$

Thus $\int_a^b f\,dx = \sup_P L(P,f)$

Upper Riemann integral is defined as the $\inf\{U(P,f): P\in P[a,b]\}$ and is denoted by $\int_a^{\bar{b}} f\,dx$.

Thus $\int_a^{\bar{b}} f\,dx = \inf_P L(P,f)$

Riemann Integral

A bound real valued function defined on $[a,b]$ is said to be Riemann integral on $[a,b]$ if its lower and upper Riemann integrals are equal i.e if

$$\int_{\underline{a}}^{b} f\,dx = \int_{a}^{\bar{b}} f\,dx.$$

The common value of these integrals is called the Riemann integral of f on $[a,b]$ and is denoted by

$$\int_{a}^{b} f\,dx$$

<u>Note</u> (1) Riemann integral is based on the notion of bounds and is subject to two conditions (i) f is bounded on the interval and
(ii) the interval is closed & finite.

(2) The family of all bounded functions which are Riemann integrable on the closed interval $[a,b]$ is denoted by $R[a,b]$

(3) f is R-integrable on $[a,b]$ $\Rightarrow$ (i) f is bounded (ii) $\int_{\underline{a}}^{b} f\,dx = \int_{a}^{\bar{b}} f\,dx = \int_{a}^{b} f\,dx$

<u>Theorem #</u> The upper and lower integrals are always defined for every bounded function on $[a,b]$

<u>Proof #</u> $\because$ f is bounded on $[a,b]$
$\therefore$ Supremum and infimum of f on $[a,b]$ exist.

Let M, m be supremum and infimum of f on $[a, b]$. Then

$$m \leq f(x) \leq M \quad \forall x \in [a, b]$$

Let $P = \{a = x_0, x_1, x_2, \ldots x_n = b\}$ be any partition of $[a, b]$ and

$$M_i = \sup f \text{ on } [x_{i-1}, x_i]$$
$$m_i = \inf f \text{ on } [x_{i-1}, x_i]$$

Then $\quad M_i \leq M \ \& \ m \leq m_i \ \forall i$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$$
$$\geq \sum_{i=1}^{n} m \Delta x_i \quad (\because m_i \geq m)$$
$$\geq m \sum_{i=1}^{n} \Delta x_i$$
$$\geq m(b-a) \longrightarrow (1)$$

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$$
$$\leq \sum_{i=1}^{n} M \Delta x_i \quad (\because M_i \leq M)$$
$$\leq M \sum_{i=1}^{n} \Delta x_i$$
$$\leq M(b-a) \longrightarrow (2)$$

Also $L(P, f) = \sum m_i \Delta x_i \leq \sum M_i \Delta x_i = U(P, f)$

$$L(P, f) \leq U(P, f) \longrightarrow (3)$$

Combining (1), (2) & (3)

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a) \quad \forall P$$

Hence the sets $\{U(P, f) : P \in P[a, b]\}$ & $\{L(P, f) : P \in P[a, b]\}$ are bounded sets and hence attain inf & sup on $[a, b]$

$\Rightarrow$ upper and lower integrals exist for a bounded function f on $[a, b]$

Theorem # If f is real valued bounded function on $[a,b]$, Then

$$\underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx$$

Proof # Let P_1, P_2 be Two partitions of $[a,b]$

Then $L(P_1, f) \leq U(P_2, f)$

This is true for each $P_1 \in P[a,b]$.

Keeping P_2 fixed, then set $\{L(P_1, f) : P_1 \in P[a,b]\}$ has an upper bound. Also

$$\sup\{L(P_1, f) : P_1 \in P[a,b]\} = \underline{\int_a^b} f\,dx$$

Since Supremum $\leq$ any upper sum

$$\therefore \underline{\int_a^b} f\,dx \leq U(P_2, f) \quad \forall\, P_2 \in P[a,b]$$

The set $\{U(P_2, f) : P_2 \in P[a,b]\}$ has a lower bound $\underline{\int_a^b} f\,dx$

But $\operatorname{Inf}\{U(P_2, f) : P_2 \in P[a,b]\} = \overline{\int_a^b} f\,dx$

Since any lower bound $\leq$ infimum.

$$\therefore \underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx$$

Theorem # If $f \in R[a,b]$, then

(i) $m(b-a) \leq \int_a^b f\,dx \leq M(b-a)$ if $b \geq a$

(ii) $m(b-a) \geq \int_a^b f\,dx \geq M(b-a)$ if $b \leq a$

where m, M are infimum and supremum of f on $[a,b]$

Proof # For $a = b$ the result is trivial

If $b \geq a$, then for every $P \in P[a,b]$, we have

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a) \quad \text{(1)}$$

Now $\sup_P L(P,f) = \underline{\int_a^b} f\,dx = \int_a^b f\,dx \quad \because f \in R[a,b]$

$\Rightarrow L(P,f) \leq \sup_P L(P,f) = \int_a^b f\,dx \longrightarrow ②$

Also $\inf_P U(P,f) = \overline{\int_a^b} f\,dx = \int_a^b f\,dx$

$\Rightarrow \inf_P U(P,f) \leq U(P,f)$

$\Rightarrow \int_a^b f\,dx \leq U(P,f) \longrightarrow ③$

From ① , ② & ③

$m(b-a) \leq L(P,f) \leq \int_a^b f\,dx \leq U(P,f) \leq M(b-a)$

$\Rightarrow m(b-a) \leq \int_a^b f\,dx \leq M(b-a)$

If $b < a$, then $a > b$

$\therefore$ interchanging in the above result

$m(a-b) \leq \int_b^a f\,dx \leq M(a-b)$

$-m(b-a) \leq -\int_a^b f\,dx \leq -M(b-a)$

$\Rightarrow m(b-a) \geq \int_a^b f\,dx \geq M(b-a)$

Darboux Theorem

If f is a bounded real function on $[a,b]$, then for each $\epsilon > 0$, $\exists\ \delta > 0$ such that

(i) $U(P,f) < \int_a^b f(x)\,dx + \epsilon$

(ii) $L(P,f) > \int_a^b f(x)\,dx - \epsilon$ for each $P \in P[a,b]$ with $\|P\| < \delta$

Proof (i) Let $\epsilon > 0$.

By definition $\overline{\int_a^b} f\,dx = \inf_P U(P,f)$

$\therefore \exists$ a partition $P_1 = \{a = x_0, x_1, x_2 \ldots x_n = b\}$ such that

$$U(P_1, f) < \overline{\int_a^b} f\,dx + \frac{\epsilon}{2} \longrightarrow (1)$$

$\because$ f is bounded on $[a, b]$

let $\sup f = M$ on $[a\ b]$

Suppose P_1 has k points other than a & b

and let $\delta = \frac{\epsilon}{4kM}$.

Let P be any partition such that $\|P\| < \delta$

and $P_2 = P_1 \cup P$

Then $U(P, f) - U(P_2, f) \leq 2kM\|P\|$

$$< 2kM\delta = 2kM \cdot \frac{\epsilon}{4kM} = \epsilon/2$$

As P_2 is a refinement of P having at most k more points than P

$\Rightarrow$ $U(P, f) < U(P_2, f) + \frac{\epsilon}{2} \leq U(P_1, f) + \epsilon/2$

$\Rightarrow$ $U(P, f) < U(P_1, f) + \epsilon/2$

$$< \overline{\int_a^b} f\,dx + \epsilon/2 + \epsilon/2 \qquad \text{by } (1)$$

$\Rightarrow$ $U(P, f) < \overline{\int_a^b} f\,dx + \epsilon$

(ii) By definition $\underline{\int_a^b} f(x)\,dx = \sup\{L(P, f) : P \in P[a, b]\}$

$\therefore$ For each $\epsilon > 0$ there exists a partition $P_1 = \{a = x_0, x_1, x_2 \ldots x_p = b\}$ such that

$$L(P_1, f) > \underline{\int_a^b} f\,dx - \epsilon/2 \longrightarrow (1)$$

Suppose P_1 has k points other than a & b

and let $\delta = \frac{\epsilon}{4kM}$, where $M = \sup f$ on $[a, b]$

Let P be a partition with $\|P\| < \delta$. Then P may contain some or none of points of P_1

If $P_2 = P \cup P_1$, Then P_2 is finer than P and contains at most k additional points.)

$$L(P_2, f) \leq L(P, f) + 2kM\|P\|$$
$$\leq L(P, f) + 2kM\delta$$

$$\Rightarrow L(P, f) + 2kM\delta \geq L(P_2, f) \geq L(P_1, f)$$

$$\Rightarrow L(P, f) + 2kM\delta \geq L(P_1, f)$$
$$> \underline{\int_a^b} f\,dx - \epsilon/2$$

$$\Rightarrow L(P, f) > \underline{\int_a^b} f\,dx - \epsilon/2 - 2kM\delta$$

$$\Rightarrow L(P, f) > \underline{\int_a^b} f\,dx - \epsilon/2 - 2kM \cdot \frac{\epsilon}{4kM}$$
$$> \underline{\int_a^b} f\,dx - \epsilon/2 - \epsilon/2 = \underline{\int_a^b} f\,dx - \epsilon$$

<u>Note</u> # The definition of infimum also leads to the fact that $U(P, f) < \overline{\int_a^b} f\,dx + \epsilon$

but this implies that for every $\epsilon > 0$ there exist at least one partition P with this property. The importance of Darboux's theorem is the existence of an infinite no of partitions P with $\|P\| < \delta$ where δ is a +ve number depending upon ϵ

The Integral as a Limit of Sums
(Riemann Sums)

The nos M_i, m_i which appear in upper and lower sums are not necessarily the values of the function f

(These are values of f if is continuous). We shall now show that $\int_a^b f\,dx$ can also be considered as the limit of a sequence of sums in which M_i, m_i are replaced by the values of f.

Corresponding to partition P of $[a,b]$, let us choose points $t_1, t_2 \ldots t_n$ such that $x_{i-1} \leq t_i \leq x_i$ and consider the sum

$$S(P,f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i$$

The sum $S(P,f)$ is called a Riemann sum of f, over $[a,b]$ relative to P

t_i are arbitrary

we say that $S(P,f)$ converges to A as $\|P\| \to 0$

i.e $$\lim_{\|P\| \to 0} S(P,f) = A$$

if, for every $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$$|S(P,f) - A| < \epsilon$$

For every partition $P = \{x_0 = a, x_1, x_2 \ldots x_n = b\}$ of $[a,b]$ with norm $\|P\| < \delta$ and for every choice of points t_i in $[x_{i-1}, x_i]$

<u>Definition</u> # A function f is said to be integrable on $[a,b]$ if $\lim S(P,f)$ exist as $\|P\| \to 0$ and then

$$\lim_{\|P\| \to 0} S(P,f) = \int_a^b f\,dx$$

For any partition P and any selection of intermediate points $t_i \in [x_{i-1}, x_i]$ $i = 1, 2 \ldots n$

$$L(P,f) \leq S(P,f) \leq U(P,f)$$

Riemann-Stieltjes Integral W.r.t An Increasing Integrator

Upper & Lower Stietjes (or Riemann Stieltjes) Sums

Let f be bounded function on $[a, b]$ and $\alpha(x)$ be defined an monotonically increasing on $[a, b]$. Corresponding to each partition. P we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

$\because$ α is monotonially increasing

$\therefore$ $\Delta\alpha_i \geq 0$

The upper and lower stieltjes or Riemann Stieltjes sum of f, w.r. α are denoted by $U(P, f, \alpha)$ and $L(P, f, \alpha)$ and are defined as

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

where

$$M_i = \operatorname{Sup}\{f(x) : x_{i-1} \leq x \leq x_i\}$$

$$m_i = \operatorname{Inf}\{f(x) : x_{i-1} \leq x \leq x_i\}$$

<u>Note</u> $\sum_{i=1}^{n} \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

$= \alpha(x_1) - \alpha(x_0) + \alpha(x_2) - \alpha(x_1) + \cdots + \alpha(x_n) - \alpha(x_{n-1})$

$= \alpha(x_n) - \alpha(x_0) = \alpha(b) - \alpha(a)$

Properties of Stieltjes Sums

1. **Theorem #** If f is bounded on $[a, b]$, α is defined and monotonically increasing on $[a. b]$, then for any partition P of $[a. b]$,

$$m[\alpha(b)-\alpha(a)] \le L(P,f,\alpha) \le U(P,f,\alpha) \le M[\alpha(b)-\alpha(a)]$$

Proof # Let $P = \{a = x_0, x_1, x_2 - - - x_n = b\}$ be a partition of $[a, b]$ and

$$M_i = \text{Sup } f(x) \quad x_{i-1} \le x \le x_i$$
$$m_i = \text{Inf } f(x) \quad x_{i-1} \le x \le x_i$$
$$M = \text{Sup } f(x) \quad x_{i-1} \le x \le x_i$$
$$m = \text{Inf } f(x) \quad x_{i-1} \le x \le x_i$$

Then

$$m \le m_i \le M_i \le M$$

$$\Rightarrow m\,\Delta\alpha_i \le m_i\,\Delta\alpha_i \le M_i\,\Delta\alpha_i \le M\,\Delta\alpha_i$$

$$\Rightarrow m\sum_{i=1}^{n}\Delta\alpha_i \le \sum m_i\,\Delta\alpha_i \le \sum_{i=1}^{n} M_i\,\Delta\alpha_i \le M\sum\Delta\alpha_i$$

$$m[\alpha(b)-\alpha(a)] \le L(P,f,\alpha) \le U(P,f,\alpha) \le M[\alpha(b)-\alpha(a)]$$

This relation holds for all partitions of $[a. b]$.

$\Rightarrow$ The sets $\{U(P,f,\alpha) : P \in P[a. b]\}$ and $\{L(P,f,\alpha) : P \in P[a,b]\}$ are bounded.

) **Theorem #** If $f : [a. b] \longrightarrow R$ be bounded on $[a. b]$ and α be defined and $\uparrow$ on $[a. b]$, Then for any partition P of $[a. b]$

(i) $L(P, f, \alpha) \le U(P, f, \alpha)$ (ii) $L(P, -f, \alpha) = -U(P, f, \alpha)$

Proof # Let $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$

and $m_i = \text{Inf } f(x) \quad x_{i-1} \leq x \leq x_i$

$M_i = \text{Sup } f(x) \quad x_{i-1} \leq x \leq x_i$

$M = \text{Sup } f(x) \quad a \leq x \leq b$

$m = \text{Inf } f(x) \quad a \leq x \leq b$

Then $m_i \leq M_i \quad \forall i$

$\Rightarrow m_i \Delta\alpha_i \leq M_i \Delta\alpha_i \quad \because \Delta\alpha_i \geq 0$

$\Rightarrow \sum_{i=1}^{n} m_i \Delta\alpha_i \leq \sum_{i=1}^{n} M_i \Delta\alpha_i$

$\Rightarrow L(P, f, \alpha) \leq U(P, f, \alpha)$

$\because f$ is bounded on $[a, b]$

$\therefore -f$ is also bounded on $[a, b]$

$\text{Sup}[-f(x)] = -\text{Inf } f(x) \quad \forall x \in [x_{i-1}, x_i]$

$= -m_i$

$\text{Inf}[-f(x)] = -\text{Sup } f(x) \quad \forall x \in [x_{i-1}, x_i]$

$= -M_i$

Hence

$U(P, -f, \alpha) = \sum_{i=1}^{n} -m_i \Delta\alpha_i = -L(P, f, \alpha)$

$L(P, -f, \alpha) = \sum_{i=1}^{n} -M_i \Delta\alpha_i = -\sum_{i=1}^{n} M_i \Delta\alpha_i$

$= -U(P, f, \alpha)$

3 Theorem # If P_1, P_2 are any Two partitions of $[a, b]$ & $P_1 \subseteq P_2$, Then

(i) $L(P_2, f, \alpha) \geq L(P_1, f, \alpha)$

(ii) $U(P_2, f, \alpha) \leq U(P_1, f, \alpha)$

(iii) $L(P_1, f, \alpha) \leq L(P_2, f, \alpha) \leq U(P_2, f, \alpha) \leq U(P_1, f, \alpha)$

(iv) $U(P_1, f, \alpha) - L(P_1, f, \alpha) \geq U(P_2, f, \alpha) - L(P_2, f, \alpha)$

<u>Proof</u> # Let $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$ and P_2 has k, additional points $c_1, c_2 \dots c_k$ such that

$$x_{i-1} < c_i < x_i \quad \forall i = 1, 2, \dots n$$

Let $P_2' = P_1 \cup \{c_1\}$ where $x_{r-1} < c_1 < x_r$, $r = 1, 2, \dots n$

Let $m_i = \text{Inf } f$ on $[x_{i-1}, x_i]$

$w_1 = \text{Inf } f$ on $[x_{r-1}, c_1]$

$w_2 = \text{Inf } f$ on $[c_1, x_r]$

Then clearly

$m_i \leq w_1$ & $m_i \leq w_2$

$$L(P_2', f, \alpha) - L(P_1, f, \alpha) = \sum_{i=1}^{r-1} m_i \Delta\alpha_i + w_1[c_1 - x_{r-1}] + w_2(x_r - c_1) + \sum_{i=r+1}^{n} m_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= \cancel{\sum_{i=1}^{r-1} m_i \Delta\alpha_i} + w_1[c_1 - x_{r-1}] + w_2[x_r - c_1] + \cancel{\sum_{i=r+1}^{n} m_i \Delta\alpha_i} - \cancel{\sum_{i=1}^{r-1} m_i \Delta\alpha_i} - m_i(x_r - x_{r-1}) - \cancel{\sum_{i=r+1}^{n} m_i \Delta\alpha_i}$$

$$= w_1(c_1 - x_{r-1}) + w_2(x_r - c_1) - m_i(x_r - x_{r-1})$$

$$= w_1(c_1 - x_{r-1}) + w_2(x_r - c_1) - m_i[(x_r - c_1) + (c_1 - x_{r-1})]$$

$= \omega_1(c_1 - x_{s-1}) + \omega_2(x_s - c_1) - m_i(x_s - c_1) - m_i(c_1 - x_{s-1})$

$= (\omega_1 - m_i)(c_1 - x_{s-1}) + \omega_2(x_s - c_1)$

$\because \omega_1 - m_i \geq 0 \quad c_1 - x_{s-1} > 0 \quad \omega_2 - m_i \geq 0$

$x_i - c_1 > 0$

$\Rightarrow L(P_2', f, \alpha) - L(P_1, f, \alpha) \geq 0$

$\Rightarrow L(P_2', f, \alpha) \geq L(P_1, f, \alpha) \longrightarrow (1)$

Let $P_2'' = \{c_2\} \cup P'$, then as above.

$L(P_2'', f, \alpha) \geq L(P_2', f, \alpha) \longrightarrow (2)$

Let $P_2''' = \{c_3\} \cup P_2''$

$L(P_2''', f, \alpha) \geq L(P_2'', f, \alpha) \longrightarrow (3)$

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Let $P_2^k = P_2^{k-1} \cup \{c_k\} = P_2$

$L(P_2, f, \alpha) \geq L(P_2^{k-1}, f, \alpha) \longrightarrow (k)$

Adding all these inequalities we get

$L(P_2, f, \alpha) \geq L(P_1, f, \alpha)$

(ii) $U(P_1, f, \alpha) - U(P_2', f, \alpha)$

$$= \sum_{i=1} M_i \Delta\alpha_i - \Big[\sum_{i=1}^{s-1} M_i \Delta\alpha_i + W_1(c_1 - x_{s-1}) + W_2(x_s - c_1) + \sum_{i=s+1}^{n} M_i \Delta\alpha_i\Big]$$

where $W_1 = \operatorname{Sup} f$ on $[x_{s-1}, c_1]$

$W_2 = \operatorname{Sup} f$ on $[c_1, x_s]$

$$= \cancel{\sum_{i=1}^{r-1} M_i \Delta\alpha_i} + M_r(x_r - x_{r-1}) + \cancel{\sum_{i=r+1}^{n} M_i \Delta\alpha_i}$$

$$- \cancel{\sum_{i=1}^{r-1} M_i \Delta\alpha_i} - W_1(c_1 - x_{r-1}) - W_2(x_r - c_1)$$

$$- \cancel{\sum_{i=r+1}^{n} M_i \Delta\alpha_i}$$

$$= M_r(x_r - x_{r-1}) - W_1(c_1 - x_{r-1}) - W_2(x_r - c_1)$$

$$= M_r(x_r - c_1 + c_1 - x_{r-1}) - W_1(c_1 - x_{r-1}) - W_2(x_r - c_1)$$

$$= (M_r - W_2)(x_r - c_1) + (M_r - W_1)(c_1 - x_{r-1}) \geq 0$$

$\because$ $M_r - W_2 \geq 0$ $\quad M_r - W_1 \geq 0$ $\quad x_r - c_1 > 0$

$c_1 - x_{r-1} > 0$

$\Rightarrow U(P_1, f, \alpha) \geq U(P_2', f, \alpha) \longrightarrow$ (1)

Let $P_2'' = P_2' \cup \{c_2\}$, Then

$U(P_2', f, \alpha) \geq U(P_2'', f, \alpha) \longrightarrow$ (2)

Let $P_2''' = P_2'' \cup \{c_3\}$, Then

$U(P_2'', f, \alpha) \geq U(P_2''', f, \alpha) \longrightarrow$ (3)

- - - - - - - - - -

- - - - - - - -

$P_2^k = P_2^{k-1} \cup \{c_k\} = P_2$, Then

$U(P_2^{k-1}, f, \alpha) \geq U(P_2^k, f, \alpha) \longrightarrow$ (k)

Adding all these inequalities, we get

$U(P_1, f, \alpha) \geq U(P_2^k, f, \alpha) = U(P_2, f, \alpha)$

$\Rightarrow U(P_1, f, \alpha) \geq U(P_2, f, \alpha)$

Proof # Let $P = P_1 \cup P_2$

Then $L(P_1, f, \alpha) \le L(P, f, \alpha)$

and $U(P, f, \alpha) \le U(P_2, f, \alpha)$

Also $L(P, f, \alpha) \le U(P, f, \alpha)$

and $L(P_2, f, \alpha) \le L(P, f, \alpha)$

$U(P, f, \alpha) \le U(P_1, f, \alpha)$

Also $L(P, f, \alpha) \le U(P, f, \alpha)$

Combining, we have.

$L(P_2, f, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P_1, f, \alpha)$

$\Rightarrow L(P_2, f, \alpha) \le U(P_1, f, \alpha)$

Again

$L(P_1, f, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P_2, f, \alpha)$

$\Rightarrow L(P_1, f, \alpha) \le U(P_2, f, \alpha)$

Theorem # Let f, g be bounded functions on $[a, b]$, α be defined & $\uparrow$ on $[a, b]$, Then for any partition P of $[a, b]$, we have

(i) $U(P, f+g) \le U(P, f) + U(P, g)$

(ii) $L(P, f+g) \ge L(P, f, \alpha) + L(P, g, \alpha)$

Proof # Let $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$ be any partition of $[a, b]$

$\because$ f, g are bounded on $[a, b]$

(iii) Now $U(P_1, f, \alpha) \geq U(P_2, f, \alpha)$

$$L(P_1, f, \alpha) \leq L(P_2, f, \alpha)$$

$$L(P_2, f, \alpha) \leq U(P_2, f, \alpha)$$

Combining

$$L(P_1, f, \alpha) \leq L(P_2, f, \alpha) \leq U(P_2, f, \alpha) \leq U(P_1,$$

(iv) $U(P_1, f, \alpha) \geq U(P_2, f, \alpha) \longrightarrow (1)$

$$L(P_1, f, \alpha) \leq L(P_2, f, \alpha)$$

$$\Rightarrow -L(P_1, f, \alpha) \geq -L(P_2, f, \alpha) \longrightarrow ($$

$(1) + (2) \Rightarrow$

$$U(P_1, f, \alpha) - L(P_1 f, \alpha) \geq U(P_2, f, \alpha) - L(P_2, f$$

OR

$$L(P_1, f, \alpha) \leq L(P_2, f, \alpha) \leq U(P_2, f, \alpha) \leq U(P_1, f$$

$$\Rightarrow U(P_1, f, \alpha) - L(P_1, f, \alpha) \geq U(P_2, f, \alpha) - L(P_2, f, \alpha)$$

[i.e on the number line distance b/w $U(P_2, f$ & $L(P_2, f, \alpha)$ can not exceed the distance bet $U(P_1, f, \alpha)$ & $L(P_1, f, \alpha)$

---

4) # Theorem # For any two partitions P of $[a \; b]$

(i) $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$

(ii) $L(P_2, f, \alpha) \leq U(P_1, f, \alpha)$

no upper sum can ever be less than any lower sum.

$\therefore f+g$ is bounded

Let $M_i = \text{Sup}(f+g)$ on $[x_{i-1}, x_i]$

$m_i = \text{Inf}(f+g)$ on $[x_{i-1}, x_i]$

$M_i' = \text{Sup} f$ on " "

$m_i' = \text{Inf} f$ on " "

$M_i'' = \text{Sup}\, g$ on " "

$m_i'' = \text{Inf}\, g$ on " "

$\because M_i', M_i''$ are supremum of f, g on $[x_{i-1}, x_i]$

$\Rightarrow f(x) \leq M_i' \quad \forall x \in [x_{i-1}, x_i]$

$g(x) \leq M_i'' \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow f(x)+g(x) \leq M_i'+M_i'' \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow (f+g)(x) \leq M_i'+M_i'' \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow M_i'+M_i''$ is an upper bound of $(f+g)$

But M_i is the least upper bound of $(f+g)$ on $[x_{i-1}, x_i]$

$$\Rightarrow M_i \leq M_i' + M_i''$$

$$\Rightarrow M_i \Delta\alpha_i \leq M_i'\Delta\alpha_i + M_i''\Delta\alpha_i$$

$$\Rightarrow \sum_{i=1}^{n} M_i\Delta\alpha_i \leq \sum_{i=1}^{n} M_i'\Delta\alpha_i + \sum_{i=1}^{n} M_i''\Delta\alpha_i$$

$$\Rightarrow U(P, f+g, \alpha) \leq U(P, f, \alpha) + U(P, g, \alpha)$$

(ii) $\because m_i', m_i''$ are the infimum of f & g on $[x_{i-1}, x_i]$

$\therefore f(x) \geq m_i' \quad \forall x \in [x_{i-1}, x_i]$

$g(x) \geq m_i''$ " " "

$f(x)+g(x) \geq m_i' + m_i'' \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow (f+g)(x) \geq m_i' + m_i''$ " "

$\Rightarrow m_i' + m_i''$ is a lower bound of $f+g$ on $[x_{i-1}, x_i]$

But m_i the greatest lower bound of $f+g$ on $[x_{i-1}, x_i]$

$\Rightarrow m_i \geq m_i'' + m_i'$

$\Rightarrow m_i \cdot \Delta\alpha_i \geq m_i' \Delta\alpha_i + m_i'' \Delta\alpha_i$

$\Rightarrow \sum_{i=1}^{n} m_i \Delta\alpha_i \geq \sum_{i=1}^{n} m_i' \Delta\alpha_i + \sum_{i=1}^{n} m_i'' \Delta\alpha_i$

$L(P, f+g) \geq L(P, f, \alpha) + L(P, g, \alpha)$

6: <u>Theorem</u> Let f be bounded on $[a, b]$, α be defined and $\uparrow$ on $[a, b]$. Then for any partition P of $[a, b]$

i) $U(P, cf, \alpha) = c\,U(P, f, \alpha)$ for any real $c \geq 0$

ii) $U(P, cf, \alpha) = c\,L(P, f, \alpha)$ if $c < 0$

iii) $|U(P, f, \alpha)| \leq U(P, |f|, \alpha)$

<u>Proof</u># (i) Let $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$ be a partition of $[a, b]$, M_i, m_i be Sup & inf of f on $[x_{i-1}, x_i]$

Then $Sup(cf) = c\, Sup\, f = c M_i$ on $[x_{i-1}, x_i]$

$M_i' = c M_i$

$U(P, cf, \alpha) = \sum_{i=1}^{n} (c M_i) \Delta\alpha_i$

$= c \sum_{i=1}^{n} M_i \Delta\alpha_i = c\,U(P, f, \alpha)$

(ii) Let $c = -c_1$ where $c_1 > 0$

$$M_i' = \operatorname{Sup}(cf) = \operatorname{Sup}(-c_1 f)$$
$$= -\operatorname{Inf}(c_1 f)$$
$$= -c_1 \operatorname{Inf} f$$
$$= c\, m_i \quad \text{on } [x_{i-1}, x_i]$$

$$U(P, cf, \alpha) = \sum M_i' \Delta\alpha_i$$
$$= \sum c\, m_i \Delta\alpha_i$$
$$= c\, L(P, f, \alpha)$$

(iii) Note $|f|(x) = |f(x)|$

$\because$ $|f| = \max(f, -f)$

$\therefore$ $f(x) \leq |f(x)| = |f|(x) \quad \forall x \in [x_{i-1}, x_i]$

$-f(x) \leq |f(x)| = |f|(x) \quad \forall x \in [x_{i-1}, x_i]$

Also $-|f(x)| \leq f(x) \leq |f(x)| = |f|(x)$

$\Rightarrow$ Let $M_i' = \operatorname{Sup}|f|$ over $[x_{i-1}, x_i]$

$$-M_i' \leq \operatorname{Sup}\{f(x) : x \in [x_{i-1}, x_i]\}$$
$$\leq \operatorname{Sup}\{|f(x)| : x \in [x_{i-1}, x_i]\} = M_i'$$

$$\Rightarrow -M_i' \leq M_i \leq M_i'$$

$$\Rightarrow -\sum M_i' \Delta\alpha_i \leq \sum_{i=1}^{n} M_i \Delta\alpha_i \leq \sum_{i=1}^{n} M_i' \Delta\alpha_i$$

$$\Rightarrow -U(P, |f|, \alpha) \leq U(P, f, \alpha) \leq U(P, |f|, \alpha)$$

$$\Rightarrow |U(P, f, \alpha)| \leq U(P, |f|, \alpha) \quad \text{Proved}$$

Riemann Stieltjes Integral

Let f real valued bounded on $[a, b]$ and α be defined and $\uparrow$ on $[a, b]$.
Let $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$ be a partition of $[a, b]$. We write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

If $M_i = \text{Sup } f$ on $[x_{i-1}, x_i]$

$m_i = \text{Inf } f$ on $[x_{i-1}, x_i]$

Then upper and Lower Riemann Stieltjes Sums are defined by

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

If m, M are infimum and supremum of f on $[a, b]$, we have

$$m \leq m_i \leq M_i \leq M$$

$$\Rightarrow m \sum \Delta\alpha_i \leq \sum m_i \Delta\alpha_i \leq \sum M_i \Delta\alpha_i \leq M \sum \Delta\alpha_i$$

$$m[\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M[\alpha(b) - \alpha(a)]$$

This inequality shows that both sets of Lower and upper sets are bound and each set Sup & inf.

The upper Riemann Stieltjes Integral w.r.t α on $[a, b]$ is defined as

$$\overline{\int_a^b} f\,d\alpha = \operatorname{Inf}\{U(P,f,\alpha) : P \in P[a,b]\}$$
$$= \operatorname{Inf}_P U(P,f,\alpha)$$

and lower Riemann Stieljes integral is defined as

$$\underline{\int_a^b} f\,d\alpha = \operatorname{Sup}\{L(P,f,\alpha) : P \in P[a,b]\}$$
$$= \operatorname{Sup}_P L(P,f,\alpha)$$

where the inf & sup are taken over all possible partitions of $[a,b]$

If $\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$

, then f is Riemann Stieltjes integrable and symbolically we write

$f \in R_\alpha[a,b]$ or $f \in (R,\ [a,b])$ or $R(\alpha)$ over $[a,b]$ or $f \in RS_\alpha$ on $[a,b]$ or $f \in RS[\alpha; a, b]$

Their common value is denoted by

$$\int_a^b f\,d\alpha \quad \text{or} \quad \int_a^b f(x)\,d\alpha(x)$$

<u>Note</u> #. Riemann Stieltjes integral is a general case of Riemann integral because if $\alpha(x) = x$, then $\Delta\alpha_i = \Delta x_i$ and the Stieltjes integral reduces to Riemann integral. In other words Riemann integral is a special case of Stieltjes integral. Since $\int_a^b f(x)\,d\alpha(x)$ depends upon f, α, a, b but not on x we write $\int f d\alpha$

letter x is a "dummy variable"

<u>Example #(a)</u> If $\alpha(x) = k \; \forall x \in [a, b]$ & f is any bounded function on $[a, b]$, Then

$$f \in R(\alpha)$$

(b) # If α is an increasing function on $[a, b]$ and $f(x) = k$, then prove that $f \in R(\alpha)$

<u>Sol</u> # Let P be any partition of $[a, b]$

Then $\alpha(x) = k \quad \forall x \in [x_{i-1}, x_i]$

and
$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$$
$$= \sum_{i=1}^{n} m_i [\alpha(x_i) - \alpha(x_{i-1})]$$
$$= \sum_{i=1}^{n} m_i [k - k] = 0 \quad \forall P$$

$\Rightarrow$ Every lower sum is zero

Hence
$$\underline{\int_a^b} f \, d\alpha = \sup_P L(P, f, \alpha) = 0$$

Again
$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i = 0 \quad \forall n$$

$$\Rightarrow \overline{\int_a^b} f \, d\alpha = 0$$

Thus
$$\underline{\int_a^b} f \, d\alpha = \overline{\int_a^b} f \, d\alpha$$

$$\Rightarrow f \in R(\alpha) \text{ on } [a, b]$$

(b) $\because f(x) = k \quad \forall x \in [a, b]$

$$\Rightarrow m_i = \inf \{ f(x) : x_{i-1} \leq x \leq x_i \}$$
$$= k \quad \forall P$$

and $M_i = k$. $\quad \forall P$

Hence $U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$

$= k[\alpha(b) - \alpha(a)] \quad \forall P$

$\overline{\int_a^b} f\,d\alpha = \inf_P U(P,f,\alpha)$

Similarly $\underline{\int_a^b} f\,d\alpha = k[\alpha(b) - \alpha(a)]$

Thus $\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$

$\Rightarrow f \in R(\alpha)$

Theorem # The upper and lower Riemann Stieltjes integrals are always defined for bounded function f

Proof # Let be bounded function on $[a,b]$ and α be defined and $\uparrow$ on $[a,b]$. Let M, m be supremum and infumum of f in $[a,b]$

Then $\quad M_i \le M \quad \forall i$

$m \le m_i \quad \forall i$

$\Rightarrow m \le m_i \le M_i \le M$

$\Rightarrow \sum m\Delta\alpha_i \le \sum m_i\Delta\alpha_i \le \sum M_i\Delta\alpha_i \; \sum M\Delta\alpha_i$

$\Rightarrow m[\alpha(b) - \alpha(a)] \le L(P,f,\alpha) \le U(P,f,\alpha) \le M[\alpha(b) - \alpha(a)]$

This is true for all partition

$\Rightarrow$ The upper and lower sums form bounded sets and attain their inf & sup

$\Rightarrow$ upper and lower integrals are always defined for bound function.

Theorem# Let f be bounded function on $[a,b]$ and α be defined and $\uparrow$ on $[a,b]$, then

$$\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha.$$

Proof# Let P_1, P_2 be any two partitions of $[a,b]$. Then

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

Keeping P_2 fixed, the set $\{L(P_1, f, \alpha) : P_1 \in P[a,b]\}$ has an upper bound. Also

$$\sup_{P_1 \in P[a,b]} L(P_1, f, \alpha) = \underline{\int_a^b} f\,d\alpha$$

$\because$ Suppremum $\leq$ any upper bound.

$$\underline{\int_a^b} f\,d\alpha \leq U(P_2, f, \alpha) \quad \forall P_2 \in P[a,b]$$

$\Rightarrow$ The set $\{U(P_2, f, \alpha) : P_2 \in P[a,b]\}$ has a lower bound $\underline{\int_a^b} f\,d\alpha$

But $$\inf_{P_2 \in P[a,b]} U(P_2, f, \alpha) = \overline{\int_a^b} f\,d\alpha$$

Since any lower bound $\leq$ Infimum

$$\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha.$$

Cauchy's Criterion for Integrability
OR
Riemann Condition

Theorem# Let f be a bound function on $[a,b]$ and α be defined, $\uparrow$ on $[a,b]$, then $f \in R(\alpha)$ iff for every $\epsilon > 0$, $\exists$ a partition P of $[a,b]$

Such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

i.e Difference b/w upper and Lower sums corresponding to any partition remain arbitrarily small

Proof# Necessary #

Let, $f \in R(\alpha)$ over $[a, b]$

Then $\int_a^b f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$

Let $\epsilon > 0$ be given.

$\because$ $\overline{\int_a^b} f\,d\alpha = \inf_P U(P, f, \alpha)$

$\therefore$ By definition of infimum $\exists$ a partition P_1 such that

$$U(P_1, f) < \overline{\int_a^b} f\,d\alpha + \frac{\epsilon}{2} \longrightarrow (1)$$

$\because$ $\underline{\int_a^b} f\,d\alpha = \sup_P L(P, f, \alpha)$

$\therefore$ By definition of supremum $\exists$ a partition P_2

$$L(P_2, f, \alpha) > \underline{\int_a^b} f\,d\alpha - \epsilon/2 \longrightarrow (2)$$

$\because$ $\int_a^b f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$

$$\Rightarrow U(P_1, f, \alpha) < \int_a^b f\,d\alpha + \epsilon/2 \longrightarrow (3)$$

$$L(P_2, f, \alpha) > \int_a^b f\,d\alpha - \epsilon/2 \longrightarrow (4)$$

Let $P = P_1 \cup P_2$. Then

$$U(P, f, \alpha) \leq U(P_1, f, \alpha) < \int_a^b f\,d\alpha + \epsilon/2$$

$$U(P,f,\alpha) < \int_a^b f\,d\alpha + \epsilon/2 \longrightarrow (5)$$

Also

$$L(P,f,\alpha) \geq L(P_2,f,\alpha) > \int_a^b f\,d\alpha - \epsilon/2$$

$$\Rightarrow L(P,f,\alpha) > \int_a^b f\,d\alpha - \epsilon/2$$

$$\Rightarrow \int_a^b f\,d\alpha - \epsilon/2 < L(P,f,\alpha) \longrightarrow (6)$$

Adding (5) & (6)

$$U(P,f,\alpha) + \cancel{\int_a^b f\,d\alpha} - \epsilon/2 < L(P,f,\alpha) + \cancel{\int_a^b f\,d\alpha} + \epsilon/2$$

$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

<u>Converse</u> (Sufficiency Condition)

Conversely suppose that for every $\epsilon > 0$ $\exists$ a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

But

$$L(P,f,\alpha) \leq \underline{\int_a^b} f\,d\alpha \longrightarrow (7)$$

&

$$\overline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha) \longrightarrow (8)$$

Adding (7) & (8)

$$L(P,f,\alpha) + \overline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha) + \underline{\int_a^b} f\,d\alpha$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

But $\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha \Rightarrow 0 \leq \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha$

$$\Rightarrow 0 \leq \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \epsilon \quad \forall \epsilon > 0$$

But a non-negative no. can be less than every +ve no. if it is zero $\therefore$ $\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha \Rightarrow f \in R(\alpha)$

Riemann Stieltjes Integral as Limit of Riemann Stieltjes Sum

Riemann Stieltjes Sum

Let be bound function f on $[a,b]$, α be be defined, $\uparrow$ on $[a,b]$. Let $P = \{a = x_0, x_1, x_2 \cdots x_n = b\}$ be a partition of $[a,b]$.
Let $T = \{t_1, t_2 \cdots t_n\}$, where $t_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \cdots n$. A sum of form

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i)\, \Delta\alpha_i$$

is called Riemann Stieltjes sum w.r.t α for the partition. It is also denoted by

$$S(P, T, f.)$$

clearly.

$$L(P, f, \alpha) \leq S(P, f, \alpha) \leq U(P, f, \alpha)$$

For any choice of points T (or t_i).

We say That

$$\lim_{\|P\| \to 0} S(P, f, \alpha) = A$$

if for every $\epsilon > 0$, $\exists\ \delta > 0$ such That
For any partition P of $[a,b]$ with $\|P\| < \delta$ and every choic of points t_i in $[x_{i-1}, x_i]$

$$|S(P, f, \alpha) - A| < \epsilon$$

Theorem # If $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ holds for some P and some ϵ, Then

(a) # It holds (with same ϵ) for every refinement of P

(b) # If $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ holds for some partition $P = \{a = x_0, x_1, x_2 - - x_n = b\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, Then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$$

Proof # (a) # Let P^* be a refinement of P

Then
$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \longrightarrow ①$$
$$U(P^*, f, \alpha) \leq U(P, f, \alpha) \longrightarrow ②$$

Adding ① & ②

$$L(P, f, \alpha) + U(P^*, f, \alpha) \leq L(P^*, f, \alpha) + U(P, f, \alpha)$$
$$U(P^*, f, \alpha) - L(P^*, f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$
$$\Rightarrow U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon$$

This result holds for any refinement of P

(b) # Let $\sup f(x) = M_i \quad \forall x \in [x_{i-1}, x_i]$

$\inf f(x) = m_i$ " " "

$\because s_i, t_i$ are arbitrary points in $[x_{i-1}, x_i]$, therefore both $f(s_i), f(t_i)$ lie in $[m_i, M_i]$ i

$$m_i \leq f(s_i) \leq M_i \longrightarrow ①$$
$$m_i \leq f(t_i) \leq M_i \longrightarrow ②$$
$$\Rightarrow |f(s_i) - f(t_i)| \leq M_i - m_i \quad \forall i$$
$$\Rightarrow \sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i \leq \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$$
$$= U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Theorem# If f is bounded function on $[a, b]$ and α be defined, $\uparrow$ on $[a\ b]$, then $f \in R(\alpha)$ iff for every real no $\epsilon > 0$ $\exists$ $\delta > 0$ (depending on ϵ) such that

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \epsilon$$

for every partition P with $\|P\| < \delta$ and for every choice $t_i \in [x_{i-1}, x_i]$

Proof# Let $f \in R(\alpha)$ and $\epsilon > 0$. Then Riemann condition holds and $\exists$ a partition P of $[a.\ b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \longrightarrow ①$$

Let P^* be a refinement of P. Then

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

and $\|P^*\| < \delta = \|P\|$

$$\Rightarrow U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon \longrightarrow ②$$

Also for the same partition P, we have.

$$L(P^*, f, \alpha) \leq \int_a^b f\,d\alpha \leq U(P^*, f, \alpha) \longrightarrow ③$$

and for the same partition P^*, we have.

$$L(P^*, f, \alpha) \leq \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \leq U(P^*, f, \alpha) \longrightarrow ④$$

By ③ & ④

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| \leq U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon$$

Hence the condition is proved

<u>Converse</u> # Suppose that for any $\epsilon > 0$ $\exists$ δ such that

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \epsilon$$

for every partition $P = \{x_0, x_1, \ldots, x_n\}$ with $\|P\| < \delta$ and for every choice of points $t_i \in [x_{i-1}, x_i]$

we prove that $f \in R(\alpha)$. Let $\alpha(b) > \alpha(a)$

$\because$ The condition holds

$\therefore$ For all choices t_i, t_i' in $[x_{i-1}, x_i]$ and for all partitions P with $\|P\| < \delta$, we have.

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \epsilon/4 \longrightarrow (5)$$

$$\left| \sum_{i=1}^{n} f(t_i')\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \epsilon/4 \longrightarrow (6)$$

Since

$M_i - m_i = \sup\{ f(x) - f(x') : x, x' \in [x_{i-1}, x_i] \}$

Therefore by definition of supremum, we have

$$M_i - m_i - \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} < f(t_i) - f(t_i')$$

$$M_i - m_i < f(t_i) - f(t_i') + \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \longrightarrow (7)$$

$U(P, f, \alpha) - L(P, f, \alpha)$

$$= \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i < \sum_{i=1}^{n} [f(t_i) - f(t_i')]\Delta\alpha_i + \frac{\epsilon}{2[\alpha(b)-\alpha(a)]} \sum_{i=1}^{n}\Delta\alpha_i$$

$$= \sum_{i=1}^{n}[f(t_i) - f(t_i')]\Delta\alpha_i + \frac{\epsilon}{2[\alpha(b)-\alpha(a)]}[\alpha(b) - \alpha(a)]$$

$$= \sum f(t_i)\Delta\alpha_i - \sum f(t_i')\Delta\alpha_i + \epsilon/2$$

$$= \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha - \sum_{i=1}^{n} f(t_i')\Delta\alpha_i + \int_a^b f d\alpha + \epsilon/2$$

$$\leq \left| \sum f(t_i)\Delta\alpha_i - \int_a^b f d\alpha - \left(\sum f(t_i')\Delta\alpha_i - \int_a^b f d\alpha\right)\right| + \epsilon/2$$

$$\leq \left| \sum f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| + \left| \sum_{i=1}^{n} f(t_i')\Delta\alpha_i - \int_a^b f d\alpha \right| + \epsilon/2$$

$$< \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon$$

$\Rightarrow$ Riemann condition is true

$\Rightarrow f \in R(\alpha)$

OR

Let the condition holds. We assume $\alpha(b) > \alpha(a)$.

Let $\epsilon > 0$; $\exists$ a no $\delta > 0$ such that

$$\left| \sum f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| < \epsilon/4 \longrightarrow (8)$$

$\forall P$, with $\|P\| < \delta$ and for every choice of points $t_i \in [x_{i-1}, x_i]$

For each i, $1 \leq i \leq n$, $\exists$ $t_i' \in [x_{i-1}, x_i]$ such that

$$M_i - \frac{\epsilon}{4[\alpha(b)-\alpha(a)]} < f(t_i')$$

$$\Rightarrow M_i < \frac{\epsilon}{4[\alpha(b)-\alpha(a)]} + f(t_i')$$

$$\Rightarrow \sum_{i=1}^{n} M_i\Delta\alpha_i < \frac{\epsilon}{4[\alpha(b)-\alpha(a)]}\sum \Delta\alpha_i + \sum_{i=1}^{n} f(t_i')\Delta\alpha_i$$

$$= \epsilon/4 + \sum_{i=1}^{n} f(t_i')\Delta\alpha_i$$

$$\Rightarrow U(P, f, \alpha) < \epsilon/4 + \sum_{i=1}^{n} f(t_i')\Delta\alpha_i \rightarrow (9)$$

Similarly for each i, $1 \leq i \leq n$ $\exists$ $t_i'' \in [x_{i-1}, x_i]$ such that

$$f(t_i'') < m_i + \frac{\epsilon}{4[\alpha(b)-\alpha(a)]}$$

$$\Rightarrow \sum_{i=1}^{n} f(t_i'')\Delta\alpha_i < \sum m_i \Delta\alpha_i + \epsilon/4$$

$$S(P,f,\alpha) < L(P,f,\alpha) + \epsilon/4 \longrightarrow (10)$$

$$\Rightarrow -L(P,f,\alpha) < -\sum f(t_i'')\Delta\alpha_i + \epsilon/4 \longrightarrow (11)$$

Adding (9) & (11)

$$U(P,f,\alpha) - L(P,f,\alpha) < \sum_{i=1}^{n} f(t_i')\Delta\alpha_i + \epsilon/4 - \sum_{i=1}^{n} f(t_i'')\Delta\alpha_i + \epsilon/4$$

$$= \sum_{i=1}^{n} f(t_i')\Delta\alpha_i - \int_a^b f\,d\alpha + \epsilon/4 + \int_a^b f\,d\alpha - \sum_{i=1}^{n} f(t_i'')\Delta\alpha_i + \epsilon/4$$

$$\leq \left|\sum_{i=1}^{n} f(t_i')\Delta\alpha_i - \int_a^b f\,d\alpha\right| + \epsilon/4 + \left|\sum_{i=1}^{n} f(t_i'')\Delta\alpha_i - \int_a^b f\,d\alpha\right| + \epsilon/4$$

$$< \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon \quad \text{by } (8)$$

$$\Rightarrow f \in R(\alpha)$$

By Muhammad Hussain
Assistant Professor
Of Maths

Properties of Riemann-Stieltjes Integra

1)# Theorem (Linearity Properties)

(a) If $f \in R(\alpha)$ on $[a,b]$, then $cf \in R(\alpha)$ for every constant c and.

$$\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha.$$

(b) If $f_1 \in R(\alpha)$, $f_2 \in R(\alpha)$, then $f_1+f_2 \in R(\alpha)$ on $[a,b]$ and

$$\int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha.$$

(c) If $f_1, f_2 \in R(\alpha)$ on $[a,b]$, then $c_1f_1+c_2f_2 \in R(\alpha)$ on $[a,b]$ and

$$\int_a^b (c_1f_1+c_2f_2)\,d\alpha = c_1\int_a^b f_1\,d\alpha + c_2\int_a^b f_2\,d\alpha$$

Proof # (a) If $c=0$, then Theorem is abious.

so let $c \neq 0$

$\because f \in R(\alpha)$

$\therefore$ Riemann Condition is True

Case I Let $c > 0$

$\because f \in R(\alpha)$

$\therefore$ For $\epsilon > 0$ $\exists$ a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon/c \rightarrow (1)$$

$$U(P,cf,\alpha) = \sum_{i=1}^{n} cM_i\Delta\alpha_i = cU(P,f,\alpha)$$

$$L(P,cf,\alpha) = \sum_{i=1}^{n} cm_i\Delta\alpha_i = cL(P,f,\alpha)$$

$$U(P,cf,\alpha)-L(P,cf,\alpha)=c[U(P,f,\alpha)-L(P,f,\alpha)]$$
$$< c\cdot \frac{\epsilon}{c} = \epsilon$$

$\Rightarrow cf \in R(\alpha)$ on $[a,b]$

Also $\int_a^b cf\,d\alpha = \overline{\int_a^b}(cf)\,d\alpha$

$$U(P,cf,\alpha) = c\,U(P,f,\alpha)$$

$$\operatorname{Inf}_P U(P,cf,\alpha) = c\operatorname{Inf}_P U(P,f,\alpha)$$

$$\overline{\int_a^b}(cf)\,d\alpha = c\overline{\int_a^b} f\,d\alpha.$$

$$\int_a^b (cf)\,d\alpha = c\int_a^b f\,d\alpha \quad \because cf \in R(\alpha)$$
$$\& \ f \in R(\alpha)$$

Case II If $c<0$

$\because f \in R(\alpha)$

$\therefore$ for $\epsilon > 0$ $\exists$ a partition P of $[a,b]$

such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\epsilon}{|c|}$$

$\operatorname{Sup}(cf) = c\operatorname{Inf} f = c\,m_i$ on $[x_{i-1}, x_i]$

$$M_i' = c\,m_i$$

$$\operatorname{Inf}(cf) = c\operatorname{Sup} f = c\,M_i$$

$$m_i' = c\,M_i$$

$$U(P,cf,\alpha) = \sum_{i=1}^{n} M_i'\Delta\alpha_i = c\sum_{i=1}^{n} m_i\Delta\alpha_i$$
$$= c\,L(P,f,\alpha)$$

$$L(P,cf,\alpha) = \sum_{i=1}^{n} m_i'\Delta\alpha_i = c\sum_{i=1}^{n} M_i\Delta\alpha_i$$
$$= c\,U(P,f,\alpha)$$

$$U(P, cf, \alpha) - L(P, cf, \alpha) = cL(P, f, \alpha) - cU(P, f, \alpha)$$
$$= -c[U(P, f, \alpha) - L(P, f, \alpha)]$$
$$= |c|[U(P, f, \alpha) - L(P, f, \alpha)]$$
$$< |c| \cdot \epsilon/|c| = \epsilon \qquad \because |c| = -c, \; c < 0$$

$\Rightarrow cf \in R(\alpha)$

$$\inf_P U(P, cf, \alpha) = \inf_P [cL(P, f, \alpha)]$$
$$= c \sup_P L(P, f, \alpha)$$
$$\overline{\int_a^b} (cf)\, d\alpha = c \int_a^b f\, d\alpha.$$

$$\Rightarrow \int_a^b (cf)\, d\alpha = c \int_a^b f\, d\alpha \qquad \because f \in R(\alpha), \; cf \in R(\alpha)$$

(b) <u>Proof</u> $\because f_1, f_2 \in R(\alpha)$

$\therefore$ For a given $\epsilon > 0$ $\exists$ partitions P_1, P_2

such That

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \epsilon/2$$
$$U(P_2, f_2, \alpha) - L(P_1, f_2, \alpha) < \epsilon/2$$

Let $P = P_1 \cup P_2$. Then

$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \epsilon/2 \longrightarrow (1)$$
$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \epsilon/2 \longrightarrow (2)$$

Let $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$

$\because f_1, f_2 \in R(\alpha)$ $\therefore f_1, f_2$ are bounded on $[a, b]$

$\Rightarrow f_1 + f_2$ is bounded on $[a, b]$

Let $f = f_1 + f_2$

$M_i = \text{Sup } f$ on $[x_{i-1}, x_i]$

$m_i = \text{Inf } f$ on $[x_{i-1}, x_i]$

$M_i' = \text{Sup } f_1$ " "

$m_i' = \text{Inf } f_1$ " "

$M_i'' = \text{Sup } f_2$ " "

$m_i'' = \text{Inf } f_2$ " "

Then $f_1(x) \le M_i' \quad \forall x \in [x_{i-1}, x_i]$

$f_2(x) \le M_i''$ " "

$\Rightarrow f_1(x) + f_2(x) \le M_i' + M_i''$ " "

$\Rightarrow (f_1 + f_2)(x) \le M_i' + M_i''$ " "

$\Rightarrow M_i' + M_i''$ is an upper bound of $f_1 + f_2$ on $[x_{i-1}, x_i]$

But M_i is least upper bound of $f_1 + f_2$ on $[x_{i-1}, x_i]$

$$M_i \le M_i' + M_i'' \qquad i = 1, 2, \dots n$$

$$\Rightarrow \sum_{i=1}^{n} M_i \Delta\alpha_i \le \sum_{i=1}^{n} M_i' \Delta\alpha_i + \sum_{i=1}^{n} M_i'' \Delta\alpha_i$$

$$U(P, f_1 + f_2, \alpha) \le U(P, f_1, \alpha) + U(P, f_2, \alpha) \rightarrow (3)$$

Similarly

$$L(P, f_1, \alpha) + L(P, f_2, \alpha) \le L(P, f_1 + f_2, \alpha) \rightarrow (4)$$

Adding (3) & (4)

$$U(P, f_1 + f_2, \alpha) + L(P, f_1, \alpha) + L(P, f_2, \alpha)$$
$$\le U(P, f_1, \alpha) + U(P, f_2, \alpha) + L(P, f_1 + f_2, \alpha)$$

$U(P, f_1+f_2, \alpha) - L(P, f_1+f_2, \alpha)$

$\leq U(P, f_1, \alpha) - L(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_2, \alpha)$

$< \epsilon/2 + \epsilon/2 = \epsilon$

$\Rightarrow f = f_1 + f_2 \in R(\alpha)$ on $[a, b]$

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha.$$

Now let $\epsilon > 0$ $\exists$ partitions P_1', P_2' of $[a, b]$ such that

$$\underline{\int_a^b} f_1(x)\, d\alpha - \epsilon/2 < L(P_1', f_1, \alpha)$$

$$\Rightarrow \int_a^b f_1\, d\alpha - \epsilon/2 < L(P_1', f_1, \alpha) \longrightarrow (5)$$

and. $$\underline{\int_a^b} f_2\, d\alpha - \epsilon/2 < L(P_2', f_2, \alpha)$$

$$\int_a^b f_2\, d\alpha - \epsilon/2 < L(P_2', f_2, \alpha) \longrightarrow (6)$$

If $P^* = P_1' \cup P_2'$, then

$L(P_1', f_1, \alpha) \leq L(P^*, f_1, \alpha)$

$L(P_2', f_2, \alpha) \leq L(P^*, f_2, \alpha)$

By (5) & (6)

$$\int_a^b f_1\, d\alpha - \epsilon/2 < L(P^*, f_1, \alpha) \longrightarrow (7)$$

$$\int_a^b f_2\, d\alpha - \epsilon/2 < L(P^*, f_2, \alpha) \longrightarrow (8)$$

Adding (7) & (8)

$$\int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha - \epsilon < L(P^*, f_1, \alpha) + L(P^*, f_2, \alpha)$$

$$\leq L(P^*, f_1+f_2, \alpha) \leq \underline{\int_a^b} (f_1+f_2)\, d\alpha$$

$\because \epsilon$ is an arbitrary

$\therefore \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha \leq \int_a^b (f_1+f_2)\,d\alpha \longrightarrow (9)$

Replacing f_1 by $-f_1$, f_2 by $-f_2$ we have from (9)

$$\int_a^b -f_1(x)\,d\alpha + \int_a^b -f_2(x)\,dx \leq \int_a^b -(f_1+f_2)\,d\alpha$$

$$\Rightarrow \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha \geq \int_a^b (f_1+f_2)\,d\alpha \longrightarrow (10)$$

By (9) & (10),

$$\int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha = \int_a^b (f_1+f_2)\,d\alpha.$$

<u>Cor</u> If $f_1, f_2 \in R(\alpha)$ on $[a,b]$, $f_1 - f_2 \in R(\alpha)$ and $\int_a^b (f_1-f_2)\,d\alpha = \int_a^b f_1\,d\alpha - \int_a^b f_2\,d\alpha$

<u>Proof</u># $\because f_2 \in R(\alpha)$

$\therefore -f_2 = (-1)f_2 \in R(\alpha)$

$\Rightarrow f_1 + (-1)f_2 \in R(\alpha)$

$\Rightarrow f_1 - f_2 \in R(\alpha)$

Also $\int_a^b (f_1 + (-1)f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b -f_2\,d\alpha$

$= \int_a^b f_1\,d\alpha - \int_a^b f_2\,d\alpha.$

(c) <u>Proof</u># $\because f_1 \in R(\alpha)$ $\quad c_1 \in R$

$\therefore c_1 f_1 \in R(\alpha)$ & $\int_a^b c_1 f_1\,d\alpha = c_1 \int_a^b f_1\,d\alpha$

Similarly $c_2 f_2 \in R(\alpha)$

and $\int_a^b c_2 f_2\,d\alpha = c_2 \int_a^b f_2\,d\alpha$

Now $c_1 f_1 \in R(\alpha)$, $c_2 f_2 \in R(\alpha)$

$\Rightarrow c_1 f_1 + c_2 f_2 \in R(\alpha)$

and $\int_a^b (c_1 f_1 + c_2 f_2) d\alpha = \int_a^b c_1 f_1 d\alpha + \int_a^b c_2 f_2 d\alpha$

$= c_1 \int_a^b f_1 d\alpha + c_2 \int_a^b f_2 d\alpha$.

<u>OR</u>

If $c_1 = 0, c_2 = 0$, then theorem is obvious.

Let $c_1 \neq 0, c_2 \neq 0$

<u>Case I</u> If $c_1 > 0, c_2 > 0$

$\because f_1 \in R(\alpha), f_2 \in R(\alpha)$

$\therefore$ For $\epsilon > 0$ $\exists$ partitions P_1, P_2 s.that

$U(P_1, f_1, \alpha) - L(P_1 f_1, \alpha) < \epsilon/2c_1$.

$U(P_2, f_1, \alpha) - L(P_2, f_2, \alpha) < \epsilon/2c_2$

Let $P = P_1 \cup P$

Then $U(P, f_1 \alpha) - L(P, f_1, \alpha) < \epsilon/2c_1$

$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \frac{\epsilon}{2c_1}$.

Let $f = c_1 f_1 + c_2 f_2$

$\text{Sup } f = \text{Sup}(c_1 f_1 + c_2 f_2)$ on $[x_{i-1}, x_i]$

Let $M_i = \sup f$ $\quad m_i = \text{Inf } f$ on $[x_{i-1}, x_i]$

$M_i' = \sup f_1$ $\quad m_i' = \text{Inf } f_1$ on $[x_{i-1}, x_i]$

$M_i'' = \sup f_2$ $\quad m_i'' = \text{Inf } f_2$ on $[x_{i-1}, x_i]$

$f_1(x) \leq M_i'$ $\quad \forall i \ \forall x \in [x_{i-1}, x_i]$

$f_2(x) \leq M_i''$ $\quad \forall i \ \forall x \in [x_{i-1}, x_i]$

$\Rightarrow c_1 f_1(x) \leq c_1 M_i'$

$c_2 f_2(x) \leq c_2 M_i''$

$\Rightarrow c_1 f_1(x) + c_2 f_2(x) \leq c_1 M_i' + c_2 M_i''$

$\Rightarrow M_i \leq c_1 M_i' + c_2 M_i''$

$\Rightarrow U(P, f, \alpha) \leq c_1 U(P, f_1 \alpha) + c_2 U(P, f_2, \alpha) \longrightarrow$ ①

Similarly

$$L(P, f, \alpha) \geq L(P, c_1 f_1, \alpha) + L(P, c_2 f_2, \alpha)$$

$$L(P, c_1 f_1, \alpha) + L(P, c_2 f_2, \alpha) \leq L(P, f, \alpha) \rightarrow (11)$$

Adding (10) & (11) &

$$U(P, f, \alpha) - L(P, f, \alpha) = c_1[U(P, f_1, \alpha) - L(P, f_1, \alpha)] + c_2[U(P, f_2, \alpha) - L(P, f_2, \alpha)]$$

$$< c_1 \frac{\epsilon}{2c_1} + c_2 \cdot \frac{\epsilon}{2c_2} = \epsilon$$

$\Rightarrow f = c_1 f_1 + c_2 f_2 \in R(\alpha)$ on $[a, b]$

Let $\epsilon > 0$ $\exists$ partitions P_1', P_2'' s. that

$$\int_a^b c_1 f_1 \, d\alpha - \epsilon/2 < L(P_1', cf_1, \alpha) \rightarrow (12)$$

$$\int_a^b c_1 f_1 \, d\alpha - \epsilon/2 < L(P_1', cf_1, \alpha) \rightarrow (12)$$

$$\& \int_a^b c_2 f_2 \, d\alpha - \epsilon/2 < L(P_2', cf_2, \alpha) \rightarrow (13)$$

Let $P^* = P_1' \cup P_2'$, Then

$$\int_a^b c_1 f_1 \, d\alpha - \epsilon/2 < L(P_1, cf_1, \alpha) < L(P^*, c_1 f_1, \alpha) \rightarrow (14)$$

$$\& \int_a^b c_2 f_2 \, d\alpha - \epsilon/2 < L(P^*, c_2 f_2, \alpha) \rightarrow (15)$$

Adding

$$\int_a^b c_1 f_1 \, d\alpha + \int_a^b c_2 f_2 \, d\alpha - \epsilon < L(P^*, c_1 f_1, \alpha) + L(P^*, c_2 f_2, \alpha)$$

$$\leq L(P^*, f, \alpha) \leq \int_a^b f \, d\alpha$$

$\because$ ϵ is an arbitrary no

$$\int_a^b c_1 f_1 \, d\alpha + \int_a^b f_2 \, d\alpha \leq \int_a^b f \, d\alpha$$

changing f_1, f_2, f -ve

$$\int_a^b c_1 f_1 \, d\alpha + \int_a^b f_2 \, d\alpha \geq \int_a^b f \, d\alpha$$

$$\Rightarrow \int_a^b c_1 f_1 \, d\alpha + \int_a^b c_2 f_2 \, d\alpha = \int_a^b (c_1 f_1 + c_2 f_2) \, d\alpha$$

$$\Rightarrow c_1 \int_a^b f_1 \, d\alpha + c_2 \int_a^b f_2 \, d\alpha = \int_a^b (c_1 f_1 + c_2 f_2) \, d\alpha.$$

<u>case II</u> If $c_1 < 0$, $c_2 < 0$

$\because$ $f_1, f_2 \in R(\alpha)$

$\therefore$ for $\epsilon > 0$ $\exists$ P_1', P_2' such that

$$U(P_1', f_1, \alpha) - L(P_1', f_1, \alpha) < \frac{\epsilon}{2|c_1|}$$

$$U(P_2', f_2, \alpha) - L(P_2', f_2, \alpha) < \frac{\epsilon}{2|c_2|}$$

Let $P^* = P_1' \cup P_2'$

$$\Rightarrow U(P^*, f_1, \alpha) - L(P^*, f_1, \alpha) < \frac{\epsilon}{2|c_1|}$$

$$U(P^*, f_2, \alpha) - L(P^*, f_2, \alpha) < \frac{\epsilon}{2|c_2|}$$

Let $f = c_1 f_1 + c_2 f_2$

$\sup f = M_i$ on $[x_{i-1}, x_i]$, $\inf f = m_i$

$\sup c_1 f_1 = c_1 \inf f_1 = c_1 m_i'$

$\inf c_1 f_1 = c_1 \sup f_1 = c_1 M_i'$

$\sup c_2 f_2 = c_2 \inf f_2 = c_2 m_i''$

$\inf c_2 f_2 = c_2 \sup f_2 = c_2 M_i''$

Now $c_1 f_1(x) \le c_1 m_i'$ $\forall x \in [x_{i-1}, x_i]$

$c_2 f_2(x) \le c_2 m_i''$

$$\Rightarrow c_1 f_1(x) + c_2 f_2(x) \le c_1 m_i' + c_2 m_i''$$

$$\Rightarrow f(x) \le c_1 m_i' + c_2 m_i''$$

$$\Rightarrow M_i \le c_1 m_i' + c_2 m_i''$$

$$\Rightarrow U(P^*, f, \alpha) \le c_1 L(P^*, f_1, \alpha) + c_2 L(P^*, f_2, \alpha) \rightarrow (1)$$

Similarly

$$L(P^*, f, \alpha) \ge c_1 U(P^*, f_1, \alpha) + c_2 U(P^*, f_2, \alpha) \rightarrow (2)$$

So $c_1 U(P^*, f_1, \alpha) + c_2 U(P^*, f_2, \alpha) < L(P^*, f, \alpha) \rightarrow (17)$

Adding (16) & (17)

$$U(P^*, f, \alpha) + c_1 U(P^*, f_1, \alpha) + c_2 U(P^*, f_2, \alpha)$$
$$< L(P^*, f, \alpha) + c_1 L(P^*, f_1, \alpha) + c_2 L(P^*, f_2, \alpha)$$

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) < -c_1 [U(P^*, f_1, \alpha) - L(P^*, f_1, \alpha)]$$
$$- c_2 [U(P^*, f_2, \alpha) - L(P^*, f_2, \alpha)]$$

$$= |c_1| [U(P^*, f_1, \alpha) - L(P^*, f_1, \alpha)]$$
$$+ |c_2| [U(P^*, f_2, \alpha) - L(P^*, f_2, \alpha)]$$
$$\leq |c_1| \frac{\epsilon}{2|c_1|} + |c_2| \frac{\epsilon}{2|c_2|} = \epsilon$$

$\Rightarrow f \in R(\alpha)$

Now it can be proved as above.

$$\int_a^b (c_1 f_1 + c_2 f_2) \, d\alpha = c_1 \int_a^b f_1 \, d\alpha + c_2 \int_a^b f_2 \, d\alpha$$

Case III If one of c_1 & c_2 is -ve say $c_1 < 0$, $c_2 > 0$

$\because f_1, f_2 \in R(\alpha)$

$\therefore$ for $\epsilon > 0$ $\exists$ partitions P_1, P_2 such that

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\epsilon}{2|c_1|}$$
$$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{\epsilon}{2c_2}$$

Let $P = P_1 \cup P_2$

Then
$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \frac{\epsilon}{2|c_1|}$$
$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \frac{\epsilon}{2c_2}$$

$\operatorname{Sup} c_1 f_1 = c_1 \operatorname{Inf} f_1 = c_1 m_i'$

$\operatorname{Inf} c_1 f_1 = c_1 \operatorname{Sup} f_1 = c_1 M_i'$

$\operatorname{Sup} c_2 f_2 = c_2 M_i''$

$\operatorname{Inf} c_2 f_2 = c_2 m_i''$

$$c_1 f_1(x) \le c_1 m_i' \qquad \forall x \in [x_{i-1}, x_i]$$

$$c_2 f_2(x) \le c_2 M_i'' \qquad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow c_1 f_1(x) + c_2 f_2(x) \le c_1 m_i' + c_2 M_i''$$

$$\Rightarrow M_i \le c_1 m_i' + c_2 M_i''$$

$$U(P, f, \alpha) \le c_1 L(P, f, \alpha) + c_2 U(P, f_2, \alpha)$$

$$c_1 M_i' \le c_1 f_1(x) \qquad \forall x \in [x_{i-1}, x_i]$$

$$c_2 m_i'' \le c_2 f_2(x) \qquad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow \cancel{m_i} \quad \cancel{\le}$$

$$c_1 M_i' + c_2 m_i'' \le c_1 f_1(x) + c_2 f(x)$$

$$c_1 M_i' + c_2 m_i'' \le m_i$$

$$\Rightarrow c_1 U(P, f_1, \alpha) + c_2 L(P, f_2, \alpha) \le L(P, f, \alpha)$$

Adding

$$U(P, f, \alpha) - L(P, f, \alpha) = -c_1 [U(P, f_1, \alpha) - L(P, f_1, \alpha)] + c_2 [U(P, f_2, \alpha) - L(P, f_2, \alpha)]$$

$$= |c_1| [U(P, f_1, \alpha) - L(P, f_1, \alpha)] + c_2 [U(P, f_2, \alpha) - L(P, f_2, \alpha)]$$

$$< |c_1| \cdot \frac{\epsilon}{2|c_1|} + c_2 \cdot \frac{\epsilon}{2c_2} = \epsilon$$

$$\Rightarrow f = c_1 f_1 + c_2 f_2 \in R(\alpha)$$

2)) # Order Preserving Properties

Under the assumption that α is monotone increasing, several useful order preserving properties of the integral can be proved. We prove the following properties

(a) If $f_1 \in R(\alpha)$, $f_2 \in R(\alpha)$ & $f_1(x) \leq f_2(x)$ on $[a, b]$, then

$$\int_a^b f_1 \, d\alpha \leq \int_a^b f_2 \, d\alpha.$$

(b) If $f \in R(\alpha)$, $f(x) \geq 0$ $\forall x \in [a, b]$, then

$$\int_a^b f(x) \, d\alpha \geq 0$$

(c) If $f \in R(\alpha)$, then $|f| \in R(\alpha)$ and

$$\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha.$$

(d) If $f \in R(\alpha)$ on $[a, b]$ & $|f(x)| \leq M$ on $[a, b]$, then

$$\left| \int_a^b f \, d\alpha \right| \leq M[\alpha(b) - \alpha(a)]$$

<u>Proof</u> # (a) Let $f(x) \geq 0$ $\forall x \in [a, b]$

Then $M_i \geq 0$

$$\Rightarrow U(P, f, \alpha) \geq 0$$

$$\Rightarrow \int_a^b f \, d\alpha \geq 0$$

$\because f_1 \leq f_2 \quad \therefore f_2 - f_1 \geq 0$

$$\Rightarrow \int_a^b (f_2 - f_1) \, d\alpha \geq 0$$

$$\Rightarrow \int_a^b f_2 \, d\alpha - \int_a^b f_1 \, d\alpha \geq 0$$

$$\Rightarrow \int_a^b f_1 \, d\alpha \leq \int_a^b f_2 \, d\alpha$$

<u>OR</u>

$\because f_1, f_2 \in R(\alpha)$

$$\therefore \underline{\int_a^b} f_1 \, d\alpha = \overline{\int_a^b} f_1 \, d\alpha = \int_a^b f_1 \, d\alpha$$

and $\underline{\int_a^b} f_2\, d\alpha = \overline{\int_a^b} f_2\, d\alpha = \int_a^b f_2\, d\alpha$. 63

Let $\text{Inf } f_1 = m_i'$ on $[x_{i-1}, x_i]$

$\text{Inf } f_2 = m_i''$ on $[x_{i-1}, x_i]$

$\because f_1(x) \leq f_2(x) \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow m_i' \leq m_i''$

$\Rightarrow \sum m_i' \Delta\alpha_i \leq \sum m_i'' \Delta\alpha_i$

$L(P, f_1, \alpha) \leq L(P, f_2, \alpha) \quad \forall P$

$\Rightarrow \underline{\int_a^b} f_1\, d\alpha \leq \underline{\int_a^b} f_2\, d\alpha$

$\Rightarrow \int_a^b f_1\, d\alpha \leq \int_a^b f_2\, d\alpha$

(c) # $\because f \in R(\alpha)$, f is bounded on $[a, b]$

$\therefore \exists$ a +ve no k, such that

$|f(x)| \leq k \Rightarrow |f|(x) \leq k \quad \forall x \in [a, b]$

$\Rightarrow |f|$ is bounded

Let P be any partition of $[a, b]$

$M_i = \text{Sup } f$ on $[x_{i-1}, x_i]$

$m_i = \text{Inf } f$ " "

$M_i' = \text{Sup } |f|$ " "

$m_i' = \text{Inf } |f|$ " "

$$\big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)| \quad \forall x, y \in [x_{i-1}, x_i]$$

$$\Rightarrow \big||f(x)| - |f(y)|\big| \leq M_i - m_i \longrightarrow (1)$$

$\because M_i', m_i'$ denot supremum and infimum on $[x_{i-1}, x_i]$

$\therefore \ f(x) \leq M_i'$ & $|f(x)| \geq m_i'$ $\forall x \in [x_{i-1}, x_i]$

$\Rightarrow \exists \ \epsilon > 0$ such that

$$|f(x)| > M_i' - \epsilon \longrightarrow ②$$

$$\& \ |f(y)| < m_i' + \epsilon \longrightarrow ③$$

from ② & ③

$$2\epsilon + |f(x)| - |f(y)| > M_i' - m_i'$$

$$M_i' - m_i' < 2\epsilon + |f(x)| - |f(y)|$$

$\because \epsilon$ is arbitrary.

$$\therefore \ M_i' - m_i' \leq |f(x)| - |f(y)| \longrightarrow ④$$

Interchang x & y

$$M_i' - m_i' \leq -(|f(x)| - |f(y)|) \longrightarrow ⑤$$

from ④ & ⑤

$$M_i' - m_i' \leq \big||f(x)| - |f(y)|\big| \longrightarrow ⑥$$

from ① & ⑥

$$M_i' - m_i' \leq M_i - m_i$$

$$\Rightarrow \sum_{i=1}^{n} (M_i' - m_i')\Delta\alpha_i \leq \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$\Rightarrow U(P, |f|, \alpha) - L(P, |f|, \alpha) \leq \underbrace{U(P, f, \alpha) - L(P, f, \alpha)}_{< \epsilon}$$

$\Rightarrow |f| \in R(\alpha)$

Since $|f| = \max\{f, -f\}$

$\therefore$

$$f(x) \le |f(x)| = |f|(x) \qquad \forall x \in [a,b]$$

$$-f(x) \le |f(x)| = |f|(x) \qquad \forall x \in [a,b]$$

$$\Rightarrow \int_a^b f(x)\,d\alpha \le \int_a^b |f|(x)\,d\alpha. \longrightarrow (7)$$

$$\& \quad \int_a^b -f(x)\,d\alpha \le \int_a^b |f|(x)\,d\alpha.$$

$$-\int_a^b f(x)\,d\alpha \le \int_a^b |f|(x)\,d\alpha.$$

$$\Rightarrow \boxed{\cancel{-\int_a^b |f|(x)\,d\alpha \ge \int_a^b f(x)\,d\alpha}}$$

$$\int_a^b f(x)\,d\alpha \ge -\int_a^b |f|(x)\,d\alpha. \longrightarrow (8)$$

By (7) & (8) we get

$$\left| \int_a^b f\,d\alpha \right| \le \int_a^b |f|(x)\,d\alpha.$$

Remarks # The Converse of the above result is not true i.e if $|f|$ is integrable, then f may or may not be integrable e.g if we take

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$$

and $\alpha(x) = x$ $\quad \cancel{\int_a^b f\,d\alpha = (b-a)}$

Then $\int_a^b f\,d\alpha = b-a$

So that f is not integrable.

But since $|f(x)| = 1 \quad \forall x$

$\int_a^b |f(x)|\,d\alpha = b-a$ i.e $|f|$ integrable

(d) # We know that

$$\int_a^b f\,d\alpha \leq U(P,f,\alpha)$$

$$= \sum M_i \Delta\alpha_i \leq M \sum \Delta\alpha_i$$

$$\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f(x)|\,d\alpha$$

$$\leq M\int_a^b d\alpha = M[\alpha(b)-\alpha(a)]$$

$$\Rightarrow \left|\int_a^b f\,d\alpha\right| \leq M[\alpha(b)-\alpha(a)]$$

3) # **Theorem** # If $f \in R(\alpha)$, then $f^2 \in R(\alpha)$ on $[a,b]$

Proof # $\because f \in R(\alpha)$ $\therefore |f| \in R(\alpha)$

$\because$ f is bounded on $[a,b]$

$\therefore$ $|f|$ is bounded on $[a,b]$

Also $f^2 = |f|^2$ is bounded on $[a,b]$

If M_i, m_i sup and inf of f on $[x_{i-1}, x_i]$, then M_i^2 & m_i^2 are sup and inf of f^2 on $[x_{i-1}, x_i]$

$\because$ f is bounded on $[a,b]$

$\therefore$ $\exists$ no M such that

$$|f(x)| \leq M \quad \& \quad M_i \leq M,\ m_i \leq M$$

Let $\epsilon > 0$

$\because$ $f \in R(\alpha)$ on $[a,b]$

$\therefore$ $\exists$ a partion P of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\epsilon}{2M} \longrightarrow (1)$$

$$U(P_1, f, \alpha) - L(P_1, f, \alpha) < \frac{\epsilon}{2k} \longrightarrow (2)$$

& $$U(P_2, f, \alpha) - L(P_2, f, \alpha) < \frac{\epsilon}{2k} \longrightarrow (3)$$

If $P = P_1 \cup P_2$, then

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon/2k \longrightarrow (4)$$

$$U(P, g, \alpha) - L(P, g, \alpha) < \frac{\epsilon}{2k} \longrightarrow (5)$$

Now for the same partition P Condition (1) holds & we have.

$$U(P, fg, \alpha) - L(P, fg, \alpha)$$

$$= \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha_i$$

$$\leq \sum_{i=1}^{n} [k(M_i' - m_i') + k(M_i'' - m_i'')] \Delta \alpha_i$$

$$= k[U(P, f, \alpha) - L(P, f, \alpha)] + k[U(P, g, \alpha) - L(P, g, \alpha)]$$

$$< k \cdot \frac{\epsilon}{2k} + k\left(\frac{\epsilon}{2k}\right) = \frac{\epsilon}{2} + \epsilon/2 = \epsilon$$

$$\Rightarrow U(P, fg, \alpha) - L(P, fg, \alpha) < \epsilon$$

$$\Rightarrow fg \in R(\alpha) \text{ on } [a, b]$$

OR

$\because f \in R(\alpha)$, $g \in R(\alpha)$

$\Rightarrow f+g \in R(\alpha)$, $f-g \in R(\alpha)$ $\qquad \because f \in R(\alpha)$

$\Rightarrow (f+g)^2 \in R(\alpha)$, $(f-g)^2 \in R(\alpha)$ $\qquad \Rightarrow f^2 \in R(\alpha)$

$\Rightarrow (f+g)^2 - (f-g)^2 \in R(\alpha)$

$\Rightarrow \frac{1}{4}[(f+g)^2 - (f-g)^2] \in R(\alpha) \Rightarrow fg \in R(\alpha)$

(c) $\because \ |f(x)| \geq t \quad \forall x \in [a,b]$

$\Rightarrow \dfrac{1}{|f(x)|} \leq \dfrac{1}{t} \quad \forall x \in [a,b]$

$\Rightarrow \left|\dfrac{1}{f(x)}\right| \leq \dfrac{1}{t}$ " "

$\Rightarrow \left|\dfrac{1}{f}(x)\right| \leq \dfrac{1}{t}$ " "

$\Rightarrow \dfrac{1}{f}$ is bound

Let $M_i' = \text{Sup}\left(\frac{1}{f}\right)$ on $[x_{i-1}, x_i]$

$m_i' = \text{Inf}\left(\frac{1}{f}\right)$ " " "

$M_i = \text{Sup}\left(\frac{1}{f}\right)$ " " "

$m_i = \text{Inf}\left(\frac{1}{f}\right)$ " " "

$\forall\ x, y \in [x_{i-1}, x_i]$

$$\left|\left(\frac{1}{f}\right)(y) - \left(\frac{1}{f}\right)(x)\right| = \frac{|f(y) - f(x)|}{|f(x)|\,|f(y)|}$$

$$\leq \frac{1}{t^2}\,|f(y) - f(x)|$$

$\Rightarrow M_i' - m_i' \leq \dfrac{1}{t^2}(M_i - m_i) \quad \forall$ partitions $\longrightarrow (1)$

$\because f \in R(\alpha)$

$\therefore$ for $\epsilon > 0$ $\exists$ a partition P of $[a, b]$

Such That,

$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon t^2$

$\sum (M_i - m_i)\Delta\alpha_i < \epsilon t^2 \longrightarrow (2)$

Now for the same partition P, we have

$U(P, f^2; \alpha) - L(P, f^2; \alpha)$

$= \sum M_i^2 \Delta\alpha_i - \sum m_i^2 \Delta\alpha_i$

$= \sum_{i=1}^{n} (M_i^2 - m_i^2) \Delta\alpha_i$

$= \sum_{i=1}^{n} (M_i + m_i)(M_i - m_i) \Delta\alpha_i$

$\leq 2M \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$

$< 2M \cdot \frac{\epsilon}{2M} = \epsilon$

$\Rightarrow f^2 \in R(\alpha)$ on $[a, b]$

Algebraic Properties of Integral.

4) # **Theorem** # Suppose that integrator is is motone increasing on $[a, b]$, $f, g \in R_\alpha[a, b]$, Then

(a) $f^2 \in R_\alpha[a, b]$

(b) $fg \in R_\alpha[a, b]$

(c) If there is a no $t > 0$ such that $|f(x)| \geq t > 0$ i.e if f is bounded away from zero, then $\frac{1}{f} \in R(\alpha)$,

Proof (a) Already proved.

(b) $\because f, g \in R(\alpha)$

$\therefore f, g$ are bounded on $[a, b]$

$\Rightarrow \exists$ numbers k_1, k_2 such that

$|f(x)| \leq k_1 \quad \forall x \in [a, b]$

$|g(x)| \leq k_2 \quad \forall x \in [a, b]$

Let $k = \max\{k_1, k_2\}$

Then $|f(x)| \leq k \quad \forall x \in [a, b]$

$|g(x)| \leq k \quad \forall x \in [a, b]$

$\Rightarrow |(fg)(x)| = |f(x)\, g(x)|$

$= |f(x)|\, |g(x)| \leq k^2 \quad \forall x \in [a, b]$

Let $M_i = \operatorname{Sup}(fg)$ on $[x_{i-1}, x_i]$

$m_i = \operatorname{Inf}(fg)$ on $[x_{i-1}, x_i]$

$M_i' = \operatorname{Sup}(f)$ " "

$m_i' = \operatorname{Inf}(f)$ " "

$M_i'' = \operatorname{Sup}(g)$ " "

$m_i'' = \operatorname{Inf}(g)$ " "

We have $\forall x, y \in [x_{i-1}, x_i]$

$(fg)(y) - (fg)(x) = f(y)g(y) - f(x)\,g(x)$

$+ f(x)g(y) - f(x)\,g(y)$

$= g(y)[f(y) - f(x)] + f(x)[g(y) - g(x)]$

$|(fg)(y) - (fg)(x)| = |g(y)[f(y) - f(x)] + f(x)[g(y) - g(x)]|$

$\leq |g(y)|\,|f(y) - f(x)| + |f(x)|\,|g(y) - g(x)|$

$\Rightarrow$

$M_i - m_i \leq k(M_i' - m_i') + k(M_i'' - m_i'') \longrightarrow (1)$

This condition holds for all partitions of [a,

$\because f, g \in R_\alpha[a, b]$

$\therefore$ for given $\epsilon > 0$ $\exists$ partitions P_1 & P_2 such

$$M_i' - m_i' \le \frac{M_i - m_i}{t^2}$$

Now

$U(P, \frac{1}{f}, \alpha) - L(P, \frac{1}{f}, \alpha)$

$= \sum (M_i' - m_i') \Delta \alpha_i$

$\le \frac{1}{t^2} \sum (M_i - m_i) \Delta \alpha_i$

$< \frac{1}{t^2} \epsilon t^2 = \epsilon$

$\Rightarrow \frac{1}{f} \in R(\alpha)$

(d) If $f, g \in R(\alpha)$ and $\exists$ $t > 0$ such that

$|g(x)| \ge t > 0 \quad \forall x \in [a, b]$

, then $\frac{f}{g} \in R(\alpha)$ on $[a, b]$

Proof # $\because$ $f, g \in R(\alpha)$

$\Rightarrow$ f, g are bounded on $[a, b]$

$\Rightarrow \exists$ a no $k > 0$ such that

$|f(x)| \le k, \quad |g(x)| \le k \quad \forall x \in [a, b]$

$\forall x \in [a\ b]$

$$\left|\left(\frac{f}{g}\right)(x)\right| = \frac{|f(x)|}{|g(x)|} \le \frac{k}{t}$$

For any $x, y \in [x_{i-1}, x_i]$

$$\left|\left(\frac{f}{g}\right)(y) - \left(\frac{f}{g}\right)(x)\right| = \left|\frac{f(y)}{g(y)} - \frac{f(x)}{g(x)}\right|$$

$$= \left|\frac{f(y)\,g(x) - f(x)\,g(y)}{g(x)\,g(y)}\right|$$

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$$= \frac{|f(y)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(y)|}{|g(x)g(y)|}$$

$$\leq \frac{|g(x)||f(y)-f(x)| + |f(x)||g(y)-g(x)|}{|g(x)||g(y)|}$$

$$\leq \frac{k}{t^2}(M_i' - m_i') + \frac{k}{t^2}(M_i'' - m_i'')$$

$$\Rightarrow M_i - m_i \leq \frac{k}{t^2}(M_i' - m_i') + \frac{k}{t^2}(M_i'' - m_i'') \longrightarrow (1)$$

$\forall P$

$\because f, g \in R(\alpha)$

$\therefore \exists$ partitions P_1, P_2 such that

$$U(P_1, f, \alpha) - L(P_1, f, \alpha) < \frac{\epsilon t^2}{2k} \longrightarrow (2)$$

$$U(P_2, g, \alpha) - L(P_2, g, \alpha) < \frac{\epsilon t^2}{2k} \longrightarrow (3)$$

Let $P = P_1 \cup P_2$, then

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon t^2}{2k} \longrightarrow (3)$$

$$U(P, g, \alpha) - L(P, g, \alpha) < \frac{\epsilon t^2}{2k} \longrightarrow (4)$$

& for this P (1) also holds.

Now

$$U(P, \tfrac{f}{g}, \alpha) - L(P, \tfrac{f}{g}, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i$$

$$\leq \frac{k}{t^2}\sum(M_i' - m_i')\Delta\alpha + \frac{k}{t^2}\sum(M_i'' - m_i'')\Delta\alpha_i$$

$$< \frac{k}{t^2}\cdot\frac{\epsilon t^2}{2k} + \frac{k}{t^2}\cdot\frac{\epsilon t^2}{2k} = \epsilon$$

$$\Rightarrow \frac{f}{g} \in R(\alpha) \text{ on } [a, b]$$

5)) # Let f bounded function $[a,b]$, α_1, α_2 be defined, $\uparrow$ on $[a,b]$, c be a the Constant.

(a) # If $f \in R(\alpha_1)$, $f \in R(\alpha_2)$, then $f \in R(\alpha_1+\alpha_2)$ and

$$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$$

(b) # If $f \in R(\alpha_1)$, then $f \in R(c\alpha_1)$ and

$$\int_a^b f\,d(c\alpha_1) = c\int_a^b f\,d\alpha_1$$

Proof # $\because$ $f \in R(\alpha_1)$ & $f \in R(\alpha_2)$

$\therefore$ For $\epsilon > 0$ $\exists$ partitions P_1 & P_2

such that

$$U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \tfrac{1}{2}\epsilon \longrightarrow (1)$$

$$\& \; U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \tfrac{1}{2}\epsilon \longrightarrow (2)$$

Let $P = P_1 \cup P_2$

Then

$$U(P, f, \alpha_1) - L(P, f, \alpha_1) < \frac{\epsilon}{2} \longrightarrow (3)$$

$$U(P, f, \alpha_2) - L(P, f, \alpha_2) < \frac{\epsilon}{2} \longrightarrow (4)$$

Let $M_i = \sup f$ on $[x_{i-1}, x_i]$

$m_i = \inf f$ " "

Let $\alpha = \alpha_1 + \alpha_2$

$$\therefore \alpha(x) = \alpha_1(x) + \alpha_2(x)$$

$$\Delta\alpha_{1i} = \alpha_1(x_i) - \alpha_1(x_{i-1})$$

$$\Delta\alpha_{2i} = \alpha_2(x_i) - \alpha_2(x_{i-1})$$

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

$$= \alpha_1(x_i) + \alpha_2(x_i) - \alpha_1(x_{i-1}) - \alpha_2(x_{i-1})$$

$$= \Delta\alpha_{1i} + \Delta\alpha_{2i}$$

$$U(P, f, \alpha) = \sum M_i \Delta\alpha_i = \sum M_i \Delta\alpha_{1i} + \sum M_i \Delta\alpha_{2i}$$

$$= U(P,f,\alpha_1) + U(P,f,\alpha_2) \longrightarrow (5)$$

Similarly

$$L(P,f,\alpha) = L(P,f,\alpha_1) + L(P,f,\alpha_2) \longrightarrow (8)$$

$$U(P,f,\alpha) - L(P,f,\alpha)$$

$$= U(P,f,\alpha_1) + U(P,f,\alpha_2) - L(P,f,\alpha_1) - L(P,f,\alpha_2)$$

$$= U(P,f,\alpha_1) - L(P,f,\alpha_1) + U(P,f,\alpha_2) - L(P,f,\alpha_2)$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \qquad \text{by (2) \& (3)}$$

$$\Rightarrow f \in R(\alpha) = R(\alpha_1+\alpha_2)$$

$$\inf_P U(P,f,\alpha) = \overline{\int_a^b} f\,d\alpha$$

$$\le U(P,f,\alpha)$$

$$= U(P,f,\alpha_1) + U(P,f,\alpha_2)$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha_1) + U(P,f,\alpha_2) \quad \forall P$$

$$\boxed{\cancel{\le \overline{\int_a^b} f\,d\alpha_1}}$$

$$\Rightarrow \int_a^b f\,d\alpha \le \inf_P U(P,f,\alpha_1) + \inf_P U(P,f,\alpha_2)$$

$$= \overline{\int_a^b} f\,d\alpha_1 + \overline{\int_a^b} f\,d\alpha_2$$

$$= \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$$

$$\Rightarrow \int_a^b f\,d(\alpha_1+\alpha_2) \le \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 \longrightarrow (7)$$

Again $\int_a^b f\,d\alpha = \sup_P L(P,f,\alpha)$

$$\ge L(P,f,\alpha) = L(P,f,\alpha_1) + L(P,f,\alpha_2)$$

$$\Rightarrow \int_a^b f\,d\alpha \ge L(P,f,\alpha_1) + L(P,f,\alpha_2) \quad \forall P$$

$$\int_a^b f\,d\alpha \geq \sup_P L(P,f,\alpha_1) + \sup_P L(P,f,\alpha_2)$$

$$\int_a^b f\,d\alpha \geq \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$$

$$\Rightarrow \int_a^b f\,d(\alpha_1+\alpha_2) \geq \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 \longrightarrow (8)$$

By (7) & (8)

$$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$$

(b) # $\because f \in R(\alpha)$

$\therefore$ for $\epsilon > 0$ $\exists$ a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon/c \longrightarrow (1)$$

For the same partition.

$$U(P,f,c\alpha_1) = \sum_{i=1}^{n} M_i \Delta(c\alpha_1)_i$$

$$= c\sum_{i=1}^{n} M_i \Delta\alpha_{1i}$$

$$= c\,U(P,f,\alpha_1) \longrightarrow (2)$$

Similarly

$$L(P,f,c\alpha_1) = c\,L(P,f,\alpha_1) \longrightarrow (3)$$

$$U(P,f,c\alpha_1) - L(P,f,c\alpha_1)$$

$$= c\,U(P,f,\alpha_1) - c\,L(P,f,\alpha_1)$$

$$= c\,[U(P,f,\alpha_1) - L(P,f,\alpha_1)]$$

$$< c\cdot \epsilon/c = \epsilon$$

$\Rightarrow f \in R(c\alpha_1)$ on $[a,b]$

$\because U(P,f,c\alpha_1) = c\,U(P,f,\alpha_1)$

$\therefore \inf_P U(P,f,c\alpha_1) = c \inf U(P,f,\alpha_1)$

$$\int_a^{\bar{b}} f\,d(c\alpha_1) = c\int_a^{\bar{b}} f\,d\alpha_1.$$

$$\Rightarrow \int_a^b f\,d(c\alpha_1) = c\int_a^b f\,d\alpha_1 \qquad \because f\in R(\alpha_1),\ f\in R(c\alpha_1)$$

<u>Theorem</u># Let α be defined and monotonic increasing on $[a,b]$. If $f\in R(\alpha)$ on $[a,b]$, $a<c<b$, then $f\in R(\alpha)$ on $[a,c]$, $f\in R(\alpha)$ on $[c,b]$ and

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

Also for any sub-interval $[c,d]$ of $[a,b]$ $f\in R_\alpha[c,d]$.

<u>Proof</u># $\because f\in R(\alpha)$ on $[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$

$\Rightarrow f$ is bounded on $[a,c]$, $[c,b]$

$\because f\in R(\alpha)$ on $[a,b]$

$\therefore$ for a given $\epsilon>0$ $\exists$ a partition P of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon \longrightarrow (1)$$

Let $P=\{a=x_0, x_1, x_2, \ldots\ldots x_n=b\}$

and $P'=\{a=x_0, x_1, \ldots, x_k, c, x_{k+1}, \ldots x_n\}$

Let $P_1=\{a=x_0, x_1, \ldots x_k, c\} = P'\cap[a,c]$

$P_2=\{c, x_{k+1}, x_{k+2}, \ldots x_n=b\} = P'\cap[c,b]$

Then P_1 & P_2 are partitions of $[a,c]$ & $[c,b]$ such that $P' = P_1\cup P_2$

$\because P \subset P'$

$\therefore U(P', f, \alpha) - L(P', f, \alpha) < \epsilon \longrightarrow (2)$

Let $M_i = \text{Sup } f$ on $[x_{i-1}, x_i]$

$m_i = \text{Inf } f$ on $[x_{i-1}, x_i]$

$M' = \text{Sup } f$ on $[x_k, c]$

$m' = \text{Inf } f$ on $[x_k, c]$

$M'' = \text{Sup } f$ on $[c, x_{k+1}]$

$m'' = \text{Inf } f$ on $[c, x_{k+1}]$

$$U(P', f, \alpha) = \sum_{i=1}^{k} M_i \Delta\alpha_i + M'(c - x_k) + M''(x_{k+1} - c) + \sum_{i=k+2}^{n} M_i \Delta\alpha_i$$

$$= \sum_{i=1}^{k} M_i \Delta\alpha_i + M'(c - x_k) + M''(x_{k+1} - c) + \sum_{i=k+2}^{n} M_i \Delta\alpha$$

$$= U(P_1, f, \alpha) + U(P_2, f, \alpha) \longrightarrow (3)$$

Also

$$L(P', f, \alpha) = L(P_1, f, \alpha) + L(P_2, f, \alpha) \longrightarrow (4)$$

Subtracting (4) from (3)

$$[U(P_1, f, \alpha) - L(P_1, f, \alpha)] + [U(P_2, f, \alpha) - L(P_2, f, \alpha)]$$

$$= U(P', f, \alpha) - L(P', f, \alpha) < \epsilon \quad \text{by } (2) \longrightarrow (5)$$

$\because U(P_1, f, \alpha) - L(P_1, f, \alpha) \geq 0$

& $U(P_2, f, \alpha) - L(P_2, f, \alpha) \geq 0$

Therefore (5) $\Longrightarrow$

$U(P_1, f, \alpha) - L(P_1, f, \alpha) < \epsilon$

& $U(P_2, f, \alpha) - L(P_2, f, \alpha) < \epsilon$

$\Rightarrow f \in R_\alpha[a, c]$ & $f \in R_\alpha[c, b]$

Proof of $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$

$$U(P,f,\alpha) \geq U(P',f,\alpha) = U(P_1,f,\alpha) + U(P_2,f,\alpha)$$
$$\geq \bar{\int_a^c} f\,d\alpha + \bar{\int_c^b} f\,d\alpha$$
$$= \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \quad \forall P$$

$$\Rightarrow \inf_P U(P,f,\alpha) \geq \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

$$\Rightarrow \int_a^b f\,d\alpha \geq \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \longrightarrow (6)$$

Again

$$L(P,f,\alpha) \leq L(P',f,\alpha) = L(P_1,f,\alpha) + L(P_2,f,\alpha)$$
$$\leq \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha$$
$$= \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

$$\Rightarrow L(P,f,\alpha) \leq \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \quad \forall P$$

$$\Rightarrow \sup_P L(P,f,\alpha) \leq \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

$$\int_a^b f\,d\alpha \leq \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \longrightarrow (7)$$

By (6) & (7)

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

By Muhammad Hussain
Assistant Professor of
Mathematics. Govt College
Asghar Mall RWP

Lemma# If M, m are supremum and infimum of f and M', m' are sup. and inf of $|f|$ on $[a, b]$, Then

$$M' - m' \leq M - m$$

Proof# $\because$ For any two real numbers c, d we have

$$||c| - |d|| \leq |c - d|$$

Therefore for any $x', x'' \in [a, b]$, we have

$$||f(x')| - |f(x'')|| \leq |f(x') - f(x'')| \longrightarrow (A)$$

$\because$ $M = \text{Sup } f$ on $[a, b]$

& $m = \text{Inf } f$ on $[a, b]$

$\therefore$ $f(x) \leq M \quad \forall x \in [a, b]$

$f(x) \geq m \quad \forall x \in [a, b]$

$\because$ $x', x'' \in [a, b]$

$\therefore$ $f(x') \leq M \longrightarrow (1)$ & $f(x'') \geq m$

$\Rightarrow -f(x'') \leq -m \longrightarrow (2)$

(1) + (2) $\Rightarrow$

$$f(x') - f(x'') \leq M - m \longrightarrow (3)$$

Interchanging x' & x''

$$f(x'') - f(x') \leq M - m$$

$$-[f(x') - f(x'')] \leq M - m \longrightarrow (4)$$

By (3) & (4)

$$|f(x') - f(x'')| \leq M - m \longrightarrow (5)$$

using (5) in (A)

$$\left| |f(x')| - |f(x'')| \right| \leq M - m \longrightarrow (6)$$

$$M' = \sup |f| \quad \text{on } [a,b]$$

$$m' = \inf |f| \quad \text{on } [a,b]$$

$$\Rightarrow |f(x)| \leq M' \;\&\; |f(x)| \geq m' \quad \forall x \in [a,b]$$

For any $\epsilon > 0$, we can find two points $x_1, x_2 \in [a,b]$ such that

$$|f(x_1)| > M' - \epsilon$$

$$M' - \epsilon < |f(x_1)| \longrightarrow (7)$$

and $\quad |f(x_2)| < m' + \epsilon$

$$\Rightarrow |f(x_2)| - m' < \epsilon \longrightarrow (8)$$

Adding (7) & (8)

$$M' - m' - \epsilon < |f(x_1)| - |f(x_2)| + \epsilon$$

$$\Rightarrow M' - m' < |f(x_1)| - |f(x_2)| + 2\epsilon \longrightarrow (9)$$

$\because$ (9) holds for arbitrary ϵ and $\forall x_1, x_2 \in [a, b]$

$$\therefore M' - m' \leq |f(x_1)| - |f(x_2)| \quad \forall x_1, x_2 \in [a,b]$$

$$\Rightarrow M' - m' \leq |f(x')| - |f(x'')| \longrightarrow (10)$$

Interchanging x' & x''

$$M' - m' \leq -\left[|f(x')| - |f(x'')| \right] \longrightarrow (11)$$

By (10) & (11), we have

$$M' - m' \leq \left| |f(x')| - |f(x'')| \right| \longrightarrow (12)$$

By (6) & (12).

$$M' - m' \leq M - m \quad \text{(Proved)}$$

Theorem# If f is continuous on $[a, b]$ and α is a monotone increasing function, then $f \in R(\alpha)$ on $[a, b]$

Proof# $\because$ f is continuous on $[a, b]$

$\therefore$ f is uniformly continuous on $[a, b]$

$\Rightarrow$ By definition of uniform continuity for $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$$|f(x_1^*) - f(x_2^{**})| < \frac{\epsilon}{[\alpha(b) - \alpha(a)]} \longrightarrow (1)$$

whenever $|x_1^* - x_2^{**}| < \delta$

(The case $\alpha(a) = \alpha(b)$ is trivial)

Choose a partition $= \{a = x_0, x_1, x_2 \cdots x_n = b\}$ such that $||P|| < \delta$. Then $\Delta x_i < \delta$

$\because$ f is continous on $[a, b]$

$\therefore$ f is bounded on $[a, b]$ and attains its sup and on $[a, b]$

$\Rightarrow$ f attains its sup M_i, inf m_i on each sub-interva $[x_{i-1}, x_i]$

$$\Rightarrow \begin{aligned} M_i &= f(x_i'') \\ m_i &= f(x_i') \end{aligned} \qquad x_i', x_i'' \in [x_{i-1}, x_i]$$

$\because$ $\Delta x_i < \delta$ & $x_i', x_i'' \in [x_{i-1}, x_i]$

$\therefore$ By (1)

$$||f(x_i'') - f(x_i')| < \frac{\epsilon}{\alpha(b) - \alpha(a)}$$

$$\Rightarrow |M_i - m_i| < \frac{\epsilon}{\alpha(b) - \alpha(a)}$$

$$\Rightarrow M_i - m_i < \frac{\epsilon}{\alpha(b) - \alpha(a)} \longrightarrow (2)$$

$$U(P,f,\alpha) - L(P,f,\alpha)$$
$$= \sum M_i \Delta\alpha_i - \sum m_i \Delta\alpha_i$$
$$= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$
$$< \frac{\epsilon}{\alpha(b)-\alpha(a)} \sum_{i=1}^{n} \Delta\alpha_i$$
$$= \frac{\epsilon}{\alpha(b)-\alpha(a)} [\alpha(b)-\alpha(a)] = \epsilon$$
$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$
$$\Rightarrow f \in R(\alpha) \text{ on } [a,b]$$

M. Hussain Assistant Professor Govt Asghar Mall College

Theorem# If f is monotone on $[a,b]$ and α is continuous and monotone increasing on $[a,b]$, then $f \in R(\alpha)$

Proof# Let f be monotonically increasing on $[a,b]$, Then
$$f(a) \leq f(x) \leq f(b) \quad \forall x \in [a,b]$$
$\Rightarrow f$ is bounded on $[a,b]$ and $\inf f = f(a)$ and $\sup f = f(b)$. Let $\epsilon > 0$ be given.

Let $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$ be a partition $[a,b]$

$\because f$ is monotone increasing

$\therefore M_i = \sup f$ on $[x_{i-1}, x_i] = f(x_i)$

$m_i = \inf f$ on $[x_{i-1}, x_i] = f(x_{i-1})$

$\because$ α is continuous and monotone increasing on the closed interval $[a, b]$

$\therefore$ α assumes every value between bounds $\alpha(a), \alpha(b)$

Therefore we can choose $\Delta\alpha_i$ such that each $\Delta\alpha_i$ is same. i.e

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n}$$ where n is the integer

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} (M_i - m_i)$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)]$$

$< \epsilon$ for large n

Now $\frac{\alpha(b) - (\alpha(a)}{n} [f(b) - f(a)] < \epsilon \longrightarrow (1)$

$\Rightarrow \frac{\alpha(b) - (\alpha(a)}{\epsilon} [f(b) - f(a)] < n$

or $n > \frac{\alpha(b) - \alpha(a)}{\epsilon} [f(b) - f(a)] \longrightarrow (a)$

$\Rightarrow$ for a given $\epsilon > 0$ we can choose no n of points of division in partition which satisfies (a) & hence (1)

so for every $\epsilon > 0$, we can choose a partition P such that

$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ on $[a\ b]$

$\Rightarrow f \in R(\alpha)$

<u>Note</u> $f \in R(\alpha)$ i.e $\int_a^b f\,d\alpha$ exists when either

(i) f is continuous & α is monotone or

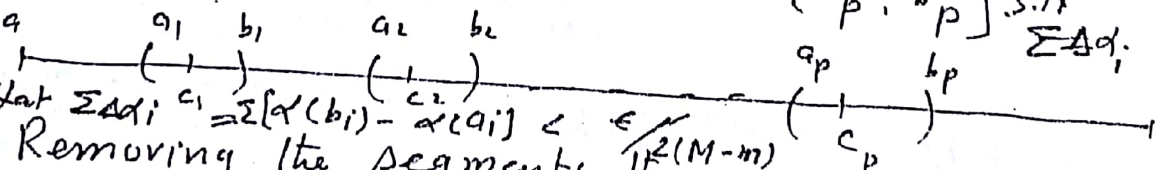
(ii) f is monotone & α is continuous & monotone

<u>Theorem</u># Let f bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$ and α is continuous at every point at which f is continuous. Then $f \in R(\alpha)$

<u>Proof</u> Let $c_1, c_2 \ldots\ldots c_p$ be finite no of points of discontinuity of f on $[a, b]$ such that $c_1 < c_2 < c_3 < \ldots\ldots < c_p$. Let $\epsilon > 0$

$\because$ set E of these points is finite

$\therefore$ we can enclose the points of E in the interior of finitely many disjoint intervals $[a_1, b_1]$ $[a_2, b_2]$ .... $[a_p, b_p]$ s.t $\Sigma\Delta\alpha_i$



that $\Sigma\Delta\alpha_i = \Sigma[\alpha(b_i) - \alpha(a_i)] < \epsilon/2(M-m)$

Removing the segments the segments (a_i, b_i) from $[a, b]$, we get a set K of remaining points i.e intervals $[a, a_1]$, $[b_1, a_2]$ ... $[b_{p-1}, a_p]$ $[b_p, b]$

$\because$ f is continuous on these intervals

$\therefore$ f is uniformly continuous on all these intervals i.e on set K.

Let $\epsilon > 0$. $\exists$ $\delta > 0$ such that

$$|f(s) - f(t)| < \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \quad \forall s, t \in K, |s-t| < \delta \longrightarrow (1)$$

Also we make disjoint intervals $[a_1, b_1]$, $[a_2, b_2]$ .... $[a_p, b_p]$ in such a way

That
$$\sum \Delta \alpha_i < \frac{\epsilon}{2(M-m)}$$ where M, m are

sup & inf of f on $[a,b]$

We form a partition $P = \{a = x_0, x_1, x_2 - - . x_n =$ of $[a.b]$ such that P contains a_i, b_i but contains no point of any segment (a_i, b_i) and if x_{i-1} is not one of a_i, then $\Delta x_i < \delta$

Now if x_{i-1} is not one of a_i i.e if $x_{i-1} \in K$, then.
$$\Delta x_i = x_i - x_{i-1} = |x_i - x_{i-1}| < \delta$$

for such intervals, we have
$$M_i - m_i < \frac{\epsilon}{2[\alpha(b) - \alpha(a)]}$$

If x_{i-1} is one of points a_i, then other end point will be some b_i and for such intervals ① is not true even though distance b/w points is less than δ because points of these intervals are not in K. For such intervals, we have.

$$m \leq m_i \leq M_i \leq M$$
$$\Rightarrow M_i - m_i \leq M - m.$$

Let A be the set of indices i for which 1st end point is not one of a_i & B be the set of those indices for which $[x_{i-1}, x_i]$ has some a_i as 1st end point.

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i\in A} (M_i - m_i)\Delta\alpha_i + \sum_{i\in B} (M_i - m_i)\Delta\alpha_i$$
$$< \frac{\epsilon}{2[\alpha(b) - \alpha(a)]}\sum \Delta\alpha_i + (M-m)\sum \Delta\alpha_i$$
$$= \frac{\epsilon}{2[\alpha(b) - \alpha(a)]}[\alpha(b) - \alpha(a)] - (M-m)\frac{\epsilon}{2(M-m)} = \epsilon$$
$$\Rightarrow f \in R(\alpha)$$

Integrability of Composition of Function

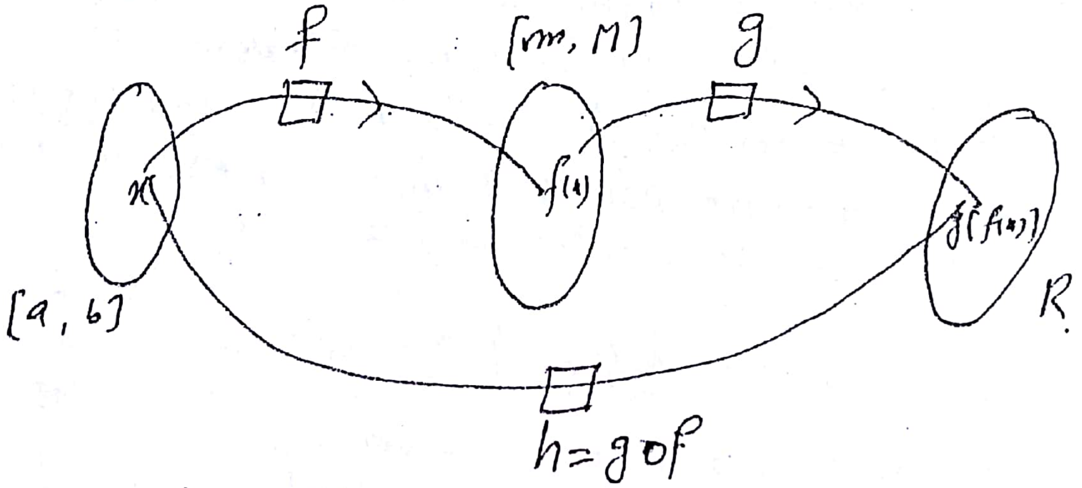
Theorem # Let f be a bounded function on $[a, b]$ and let α be monotone increasing on $[a, b]$. Suppose that $m \leq f(x) \leq M$ $\forall x \in [a, b]$. If $f \in R(\alpha)$ on $[a, b]$ and g is continuous on $[m, M]$, then $h = g\circ f \in R_\alpha[a, b]$

Proof # Let $h = g\circ f$

$\because$ A function continuous on a closed interval is bounded and g is continuous on $[m, M]$

$\therefore$ g is bounded on $[m, M]$

Let $|g(t)| \leq K$ $\quad \forall t \in [m, M]$



$\because$ g is continuous on $[m, M]$

$\therefore$ g is uniformly continuous on $[m, M]$

Therefore for $\epsilon > 0$ $\exists$ δ, $0 < \delta < \dfrac{\epsilon}{2K + \alpha(b) - \alpha(a)}$

such that

$$|g(s) - g(t)| < \frac{\epsilon}{2K + \alpha(b) - \alpha(a)} \quad \text{if } |s - t| < \delta \quad \longrightarrow (1)$$

$\because$ $f \in R(\alpha)$

$\therefore$ $\exists$ a partition $P = \{a = x_0, x_1, x_2 \cdots x_n = b\}$ such that

$U(P,f,\alpha) - L(P,f,\alpha) < \delta^2 \longrightarrow (2)$

Let $M_i = \text{Sup } f$ on $[x_{i-1}, x_i]$
$m_i = \text{Inf } f$ " " "
$M_i' = \text{Sup } f$ " " "
$m_i' = \text{Inf } f$ " " "

Dividing the idices $i = 1, 2, 3 \ldots n$ into two classes A & B as under

$$A = \{ i \mid 1 \le i \le n \text{ \& } M_i - m_i < \delta \}$$

$$B = \{ i \mid 1 \le i \le n \text{ \& } M_i - m_i \geq \delta \}$$

If $i \in A$, we have by (1)

$$M_i' - m_i' < \frac{\epsilon}{2K + \alpha(b) - \alpha(a)}$$

If $i \in B$, we have from

$$|g(t)| \le K \quad \forall t \in [a,b]$$

$$\Rightarrow -K \le g(t) \le K.$$

$$\Rightarrow 0 \le g(t) \le 2K$$

$$\Rightarrow M_i' - m_i' \le 2K.$$

Now $\delta \le M_i - m_i \quad \forall i \in B$

$$\sum_{i \in B} \delta \Delta \alpha_i \le \sum_{i \in B} (M_i - m_i) \Delta \alpha_i \le \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha$$

$$= U(P,f,\alpha) - L(P,f,\alpha) < \delta^2 \text{ by } (2)$$

$$\Rightarrow \delta \sum_{i \in B} \Delta \alpha_i < \delta^2 \Rightarrow \sum_{i \in B} \Delta \alpha_i < \delta \longrightarrow (3)$$

Therefore

$$U(P,h,\alpha) - L(P,h,\alpha) = \sum_{i=1}^{n} (M_i' - m_i') \Delta \alpha_i$$

$$= \sum_{i \in A} (M_i' - m_i') \Delta \alpha_i + \sum_{i \in B} (M_i' - m_i') \Delta \alpha_i$$

$$< \frac{\epsilon}{2K + (\alpha(b) - \alpha(a))} \sum_{i \in A} \Delta \alpha_i + 2K \sum_{i \in B} \Delta \alpha_i$$

$$< \frac{\epsilon}{2K+\alpha(b)-\alpha(a)} \sum_{i\in A} \Delta\alpha_i + 2k\delta$$

$$< \frac{\epsilon}{2K+\alpha(b)-\alpha(a)} \sum_{i=1}^{n} \Delta\alpha_i + 2k\delta \qquad \because \sum_{i\in A} \Delta\alpha_i \le \sum_{i=1}^{n} \Delta\alpha_i$$

$$= \frac{\epsilon}{2K+\alpha(b)-\alpha(a)} [\alpha(b)-\alpha(a)] + 2k \frac{\epsilon}{2K+\alpha(b)-\alpha(a)}$$

$$= \frac{\epsilon\,[2k+\alpha(b)-\alpha(a)]}{2k+\alpha(b)-\alpha(a)} = \epsilon$$

$\Rightarrow U(P,h,\alpha) - L(P,h,\alpha) < \epsilon$

$\Rightarrow h = g\circ f \in R(\alpha)$

Reduction To Riemann Integral

Theorem# Assume α increases monotonically and $\alpha' \in R$ on $[a\ b]$. Let f be bounded function on $[a\ b]$. Then $f\in R(\alpha)$ iff $f\alpha' \in R$ on $[a,b]$ and $\int_a^b f\,d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx$

Proof# $\because \alpha' \in R$ on $[a,b]$

$\therefore$ For a given $\epsilon > 0$ $\exists$ a partition $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$ of $[a,b]$ s.t

$$U(P,\alpha') - L(P,\alpha') < \epsilon \longrightarrow ①$$

$\because \alpha'$ exists on $[a,b]$

$\therefore \alpha$ is continuous on $[a,b]$ & hence on $[x_{i-1}, x_i]$ and its derivative also exists in $]x_{i-1}, x_i[$

By MVT there exists points $t_i \in]x_{i-1}, x_i[$ such that

$$\frac{\alpha(x_i) - \alpha(x_{i-1})}{x_i - x_{i-1}} = \alpha'(t_i) \quad \forall i$$

$$\Rightarrow \alpha(x_i) - \alpha(x_{i-1}) = (x_i - x_{i-1})\,\alpha'(t_i) \quad \forall i$$

$\Rightarrow \Delta\alpha_i = \Delta x_i \alpha'(t_i) \quad \forall i$

If $s_i \in [x_{i-1}, x_i]$, then $\alpha'(t_i), \alpha'(s_i)$ both lie in $[m_i, M_i]$, where

$$M_i = \sup \alpha' \text{ on } [x_{i-1}, x_i]$$

$$m_i = \inf \alpha' \text{ on } [x_{i-1}, x_i]$$

$\Rightarrow |\alpha'(s_i) - \alpha'(t_i)| \leq M_i - m_i$

$\Rightarrow \sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i \leq \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$

$< \epsilon$ by ①

$\Rightarrow \sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i < \epsilon \longrightarrow$ ②

$\because$ f is bounded on $[a, b]$

Let $|f(x)| \leq M$ where $M > 0$ $\forall x \in [a, b]$

<u>Step-II</u>

Consider

$\left| \sum_{i=1}^{n} f(s_i) \Delta\alpha_i - \sum_{i=1}^{n} f(s_i) \alpha'(s_i) \Delta x_i \right|$

$= \left| \sum_{i=1}^{n} f(s_i) \Delta\alpha_i - \sum_{i=1}^{n} f(s_i) \alpha'(s_i) \Delta x_i \right|$

$= \left| \sum_{i=1}^{n} f(s_i) \alpha'(t_i) \Delta x_i - \sum f(s_i) \alpha'(s_i) \Delta x_i \right|$ $\quad \because \Delta\alpha_i = \alpha' \Delta x_i$

$= \left| \sum_{i=1}^{n} f(s_i) [\alpha'(t_i) - \alpha'(s_i)] \Delta x_i \right|$

$\leq \left| \sum M [\alpha'(t_i) - \alpha'(s_i)] \right| \Delta x_i$

$\leq M\epsilon$ by ② $\longrightarrow$ ③

$\Rightarrow -M\epsilon < \sum f(s_i)\Delta\alpha_i - \sum f(s_i)\alpha'(s_i)\Delta x_i < M\epsilon \longrightarrow (4)$

$\Rightarrow \sum_{i=1}^{n} f(s_i)\Delta\alpha_i < \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i + M\epsilon \longrightarrow (5)$

$\forall s_i \in [x_{i-1}, x_i]$

and. $\sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i < \sum_{i=1}^{n} f(s_i)\Delta\alpha_i + M\epsilon \longrightarrow (6)$

$\forall s_i \in [x_{i-1}, x_i]$

$\Rightarrow U(P, f\alpha) \leq U(P, f\alpha') + M\epsilon \longrightarrow (7)$

and. $U(P, f\alpha') \leq U(P, f, \alpha) + M\epsilon \longrightarrow (8)$

from (7) & (8) we have.

$|U(P, f, \alpha) - U(P, f\alpha')| \leq \epsilon M \longrightarrow (9)$

This remains true if P is replaced by any refinement because (1) remains true for any refinement of P

Therefore.

$$\left| \overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\,\alpha'(x)\,dx \right| \leq M\epsilon$$

$\because$ ϵ is an arbitrary and may be very small. Therefore.

~~Similarly from (4)~~

$$\left| \overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\,\alpha'(x)\,dx \right| = 0$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f(x)\,\alpha'(x)\,dx \longrightarrow (10)$$

Similarly from (4) we can prove that

$$\underline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f(x)\,\alpha'(x)\,dx \longrightarrow (11)$$

Hence if $f \in R(\alpha)$ and $[a, b]$, Then

$$\int_a^b f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha.$$

and from (10) & (11) we have.

$$\overline{\int_a^b} f(x)\,\alpha'(x)\,dx = \int_a^b f(x)\,\alpha'(x)\,dx$$

$$\Rightarrow \quad f\alpha' \in R \text{ on } [a,b]$$

Conversely if $f\alpha' \in R$, then from (1) & (11)

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

$\Rightarrow f \in R(\alpha)$ on $[a, b]$.

Change of Variable in Riemann-Stieltjes Integral.

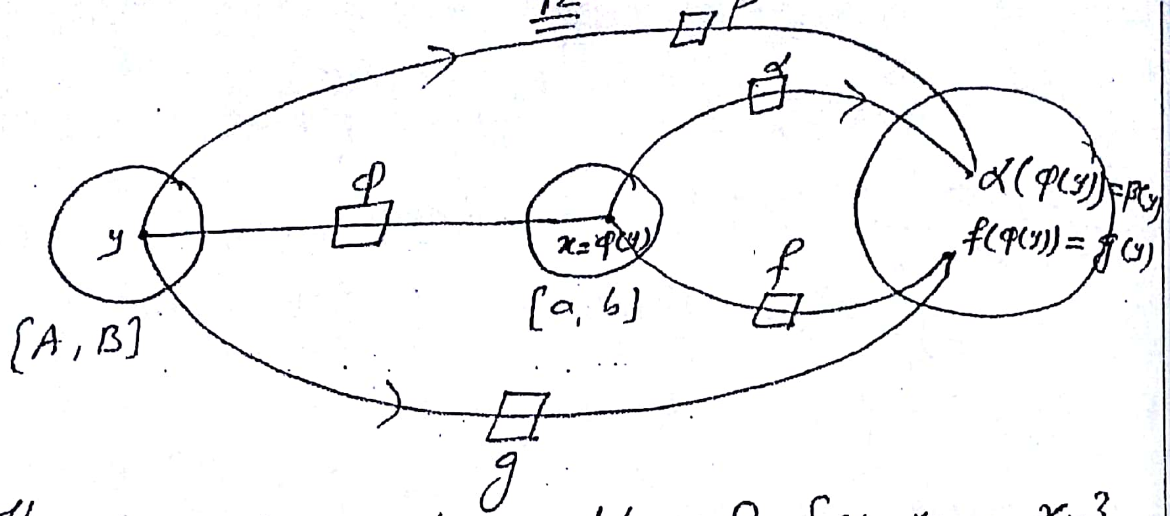
Theorem # Let $f \in R_\alpha[a,b]$ and φ be a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. α is monotonically increasing on $[a,b]$. Suppose α is monotonically increasing. Let β & g be composite function defined on $[A, B]$ as

$$\beta(y) = \alpha[\varphi(y)]$$
$$g(y) = f[\varphi(y)] \quad \forall y \in [A, B]$$

Then $g \in R(\beta)$ on $[A, B]$ and.

$$\int_A^B g\,d\beta = \int_a^b f\,d\alpha.$$

Proof # $\because$ φ is strictly increasing on $[A, B]$
$\therefore$ φ is 1-1 and has strictly increasing inverse φ^{-1} on $[a, b]$



Therefore to each partition $P = \{x_0, x_1 \dots x_n\}$ of $[a, b]$, there corresponds one and only one partition $Q = \{y_0, y_1 \dots y_n\}$ of $[A\ B]$ so that

$$x_i = \varphi(y_i)$$

All partitions of $[a, b]$ are obtained in this way

Also any refinement of Q produces a corresponding refinement of P and converse is also true.

$\because$ The values taken by f on $[x_{i-1}, x_i]$ are exactly same as those taken by g on $[y_{i-1}, y_i]$. So

$$U(Q, g, \beta) = U(P, f, \alpha) \longrightarrow ①$$

$$\& \ L(Q, g, \beta) = L(P, f, \alpha) \longrightarrow ②$$

$\because f \in R(\alpha)$ on $[a, b]$

$\therefore$ given $\epsilon > 0$, we have.

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\Rightarrow U(Q, g, \beta) - L(Q, g, \beta) < \epsilon$$

$$\Rightarrow g \in R(\beta) \text{ on } [A, B]$$

from ①
$$\inf_Q U(Q, g, \beta) = \inf_P U(P, f, \alpha)$$

$$\overline{\int_A^B} g\, d\beta = \int_A^B g\, d\beta = \overline{\int_a^b} f\, d\alpha = \int_a^b f\, d\alpha$$

Remarks # This Theorem applies in particular to Riemann integrals i.e when $\alpha(x) = x$. Another Theorem of this type in which α is not required to be monotone will later be proved for Riemann integral.

Question # Suppose that $f \geq 0$ i.e. f is non-negative, f is continuous on $[a, b]$ and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ $\forall x \in [a, b]$. Does its converse holds. Prove your assertion.

Sol # Since f is continuous on $[a, b]$, it is integrable. Suppose that there a point x_0 in $[a, b]$ where. i.e $f(x_0) \neq 0$

$$f(x_0) > 0$$

Now if a function f is continuous at point c and $f(c) \neq 0$, then f is locally bounded away from zero i.e $\exists$ a nbd $N(c)$ of c and the constant m such that

$$|f(x)| \geq m > 0 \quad \forall x \in [a, b] \cap N(c)$$

$\because$ f is continuous at x_0 & $f(x_0) \neq 0$

$\therefore$ f is bounded away from zero at x_0 and Nbd. $N(x_0)$ of x_0

i.e $\exists$ a no $K > 0$ such that $\because f(x) \geq 0$

$$0 < m \leq |f(x)| = f(x) \quad \forall x \in N(x_0) \cap [a, b]$$

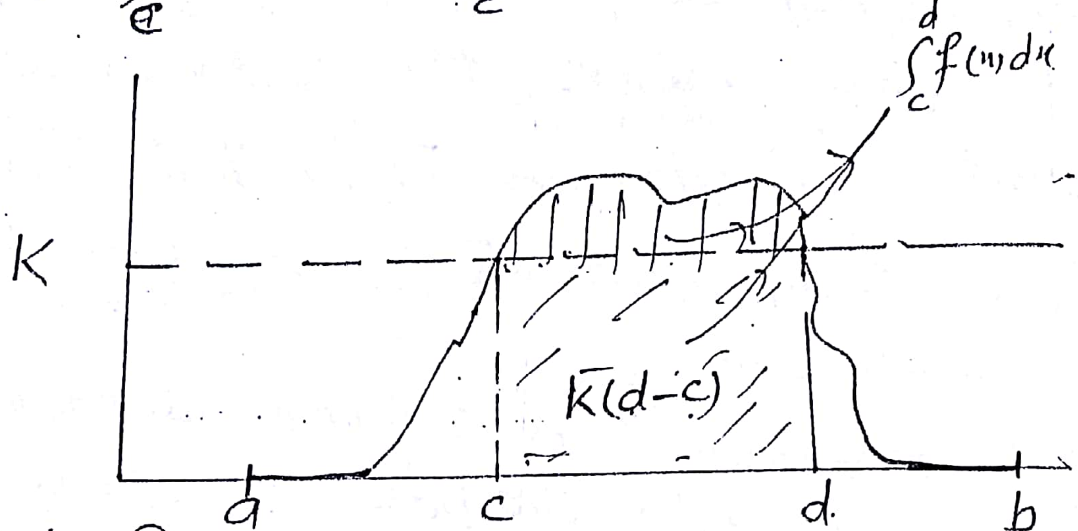
Let the interval $[a,b] \cap N(x_0)$ have end points c, d with $c<d$

$\because$ f is continuous on $[c,d]$

$\therefore$ f is integrable on $[c,d]$

Now $f(x) \geq K$ on $[c,d]$

$$\Rightarrow \int_c^d f(x)\,dx \geq \int_c^d K\,dx = K(d-c) > 0$$



But $f(x) \geq 0 \;\forall x \in [a,b]$

Therefore.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^b \geq \int_c^d f\,dx > 0$$

$$\Rightarrow \int_a^b f(x)\,d > 0$$ which contradicts the hypothesis that $\int_a^b f\,dx = 0$

Therefore there can exist no point in $[a,b]$ where f is not zero. We conclude that f is identically zero on $[a,b]$

Converse

Its Converse is als true i.e if $f(x) \geq 0$ & $f(x) = 0$, $\forall x \in [a, b]$, then

$$\int_a^b f\,dx = 0$$

$\because$ $f(x) \geq 0$ & $f(x) = 0$

$\therefore$ $U(P, f) = 0 = L(P, f)$ $\forall P$

Therefore

$$\int_a^b f\,dx = 0$$

By. M. Hussain Assistant Professor Asghar Mall College

Integration & Differentiation

Function Defined by Definite Integrals

Let f be Riemann integrable on $[a, b]$, Then The function F given by

$$F(x) = \int_a^x f(t)\,dt \quad \forall x \in [a, b]$$

is well defined because for each $x \in [a, b]$ is $f \in R[a, x]$ and as such $F(x)$ is uniquely defined on $[a, b]$. The function F may be called the integral function of f. We examine certain properties of this function F defined on $[a, b]$

<u>Note</u> we observe that $F(a) = \int_a^a f\,dt$ and $F(b) = \int_a^b f\,dt$.

Theorem# (Idefinite-integral Theorem OR 1st fundamental Theorem)

Let $f \in R[a,b]$ (or Let f be continuous on $[a,b]$). Then function F defined by

$$F(x) = \int_a^x f(t)\,dt \quad \forall x \in [a,b]$$

is continuous on $[a,b]$; further if f is continuous at a point x_0 of $[a,b]$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0)$$

Proof# $\because$ $f \in R[a,b]$

$\therefore$ f is bounded on $[a,b]$

Let $|f(t)| \leq M \quad \forall t \in [a,b]$

Let us choose any two points in $[a,b]$ such that

$$a \leq x < y \leq b$$

$$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_a^y f(t)\,dt + \int_x^a f(t)\,dt$$

$$= \int_x^a f(t)\,dt + \int_a^y f(t)\,dt$$

$$= \int_x^y f(t)\,dt$$

$$\Rightarrow |F(y) - F(x)| = \left|\int_x^y f(t)\,dt\right| \leq M(y-x)$$

For any $\epsilon > 0$ such that

$$M|(y-x)| < \epsilon \quad \text{when } |y-x| < \frac{\epsilon}{M}$$

Then $|F(y)-F(x)| < \epsilon$ whenever $|y-x| < \frac{\epsilon}{M} = \delta$

So that for any $\epsilon > 0$ we can always choose $\delta = \frac{\epsilon}{M}$ such that

$$|F(y)-F(x)| < \epsilon \quad \text{whenever } |y-x| < \delta$$

$\Rightarrow$ F is uniformly continuous on $[a, b]$

$\because$ uniformly continuous function is continuous

$\therefore$ F is continuous on $[a, b]$

Differentiability At x_0

Suppose that f is continuous at x_0.

Then by definition of continuity for $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$$|f(t)-f(x_0)| < \epsilon \quad \text{whenever } |t-x_0| < \delta \longrightarrow (1)$$

$x_0-\delta < t < x_0+\delta$

$$\Rightarrow f(x_0)-\epsilon < f(t) < f(x_0)+\epsilon$$

Now

$$\frac{F(t)-F(x_0)}{t-x_0} - f(x_0)$$

$$= \frac{1}{t-x_0} \int_{x_0}^{t} f(t)\,dt - f(x_0) \quad \text{by definition of } F$$

$$= \frac{1}{t-x_0} \left[\int_{x_0}^{t} f(t)\,dt - (t-x_0) f(x_0) \right]$$

$$= \frac{1}{t-x_0}\left[\int_{x_0}^{t} f(t)\,dt - f(x_0)\int_{x_0}^{t} dt\right]$$

$$\Rightarrow \left|\frac{F(t)-F(x_0)}{t-x_0} - f(x_0)\right| = \frac{1}{|t-x_0|}\left|\int_{x_0}^{t}[f(t)-f(x_0)]dt\right|$$

$$\leq \frac{1}{|t-x_0|}\int_{x_0}^{t}|f(t)-f(x_0)|dt \longrightarrow ②$$

From ①

$$\int_{x_0}^{t}|f(t)-f(x_0)|dt < \int_{x_0}^{t}\epsilon\,dt \quad \text{when } |t-x_0|<\delta$$

$$\int_{x_0}^{t}|f(t)-f(x_0)|dt < \epsilon|(t-x_0)| \quad |t-x_0|<\delta$$

using in ②

$$\left|\frac{F(t)-F(x_0)}{t-x_0} - f(x_0)\right| < \frac{1}{|t-x_0|}\epsilon|t-x_0| = \epsilon$$

$$\Rightarrow \left|\frac{F(t)-F(x_0)}{t-x_0} - f(x_0)\right| < \epsilon \quad \text{provided } 0<|t-x_0|<\delta$$

Hence we conclude that F is differentiable at x_0 and

$$\lim_{t\to x_0}\frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$$

$$\Rightarrow F'(x_0) = f(x_0)$$

Results # (1) We note that function
$F(x) = \int_a^x f(t)\,dt$ is always continuous
though $f(x)$ may not be continuous. Thus
the process of integration generates
continuous functions.

(2) # If f is continuous on $[a, b]$, then
F is differentiable on $[a, b]$. Thus the process
of integration applied to continuous functions
generates differentiable functions. If f is diffable
at a point, then F is twice differentiable at
that point.

(3) Every continuous function f on $[a, b]$
is derivative of a function
$$F(x) = \int_a^x f(t)\,dt$$
& $\frac{d}{dx}[F(x)] = f(x)$

$\Rightarrow$ F is a member of family of primitives
of f on $[a, b]$

The family of primitives of a continuous function
f is denoted by $\int f\,dx$ and is called
indefinite integral of f.

Primitive # If f and F are two functions
defined on $[a, b]$ and $F'(x) = f(x)$ $\forall x \in [a, b]$,
then F is called a primitive of f on $[a, b]$

Theorem # Suppose that F be a primitive of a continuous function f on $[a, b]$. A function G defined on $[a, b]$ is also a primitive of f on $[a, b]$ iff for some constant c, $G(x) = F(x) + c$ $\forall x \in [a, b]$

Proof # Suppose G is a primitive of f. Then

$$F'(x) = f(x) \quad \forall x \in [a, b]$$

$$G'(x) = f(x) \quad \forall x \in [a, b]$$

$$\Leftrightarrow \quad F'(x) = G'(x)$$

$$\Leftrightarrow \quad F'(x) - G'(x) = 0$$

$$\Leftrightarrow \quad (F' - G')(x) = 0$$

$$\Leftrightarrow \quad F - G = c$$

$$\Leftrightarrow \quad F = G + c$$

or

$$G = F - c = F + c_1$$

Note (1) Any two primitives of a function differ by a constant.

(2) Primitive is also called anti-derivative.

Cauchy's Fundamental Theorem of Calculus

OR

2nd Fundamental Theorem

If $f \in R[a,b]$ and there is a differentiable function F on $[a,b]$ such that $F'(x) = f(x) \quad \forall x \in [a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

OR

If f is continuous on $[a,b]$ and F is a primitive of f on $[a,b]$, then

$$\int_a^b f\,dx = F(b) - F(a)$$

Proof $\because f \in R[a,b]$

$\therefore$ For any $\epsilon > 0$, $\exists$ a partition $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of $[a,b]$ such that

$$U(P,f) - L(P,f) < \epsilon \quad \text{①}$$

$\because F$ is differentiable on $[a,b]$

$\therefore$ By MVT $\exists$ points $t_i \in \,]x_{i-1}, x_i[$ such that

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(t_i) = f(t_i) \quad \forall i = 1,2,\dots,n$$

$$\Rightarrow F(x_i) - F(x_{i-1}) = f(t_i)\,\Delta x_i \quad \forall i$$

Summing over all the intervals

$$\sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

$$F(b) - F(a) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

(telescoping sum)

This will be true for every partition of $[a, b]$

Now under condition ① for any t_i in $[x_{i-1}, x_i]$ we have

$$\left| \sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f dx \right| < \epsilon$$

$$\Rightarrow \left| F(b) - F(a) - \int_a^b f dx \right| < \epsilon$$

Since it holds for any arbitrary $\epsilon > 0$, therefore

$$\int_a^b f dx = F(b) - F(a)$$

Corollary # If f is differentiable on $[a, b]$ and if f' is integrable on $[a, b]$, Then

$$\int_a^b f'(x) dx = f(b) - f(a)$$

Actually we can write.

$$\int_a^x f'(x) dx = f(x) - f(a)$$

Note # The fundamental theorem does not state that if f is integrable, then f has a primitive. It only states that if f has

a primitive on $[a,b]$, then the primitive can be used to evaluate $\int_a^b f(x)dx$. The question whether a primitive exists and the question of the existence of an integral of function $f(x)$ on $[a,b]$ are entirely independent.

Counter Example

A function may have a primitive without being integrable.

Consider the function F and f defined on $[-1,1]$ as

$$F(x) = \begin{cases} x^2 \sin\frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and

$$f(x) = \begin{cases} 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

clearly $F'(x) = f(x)$

$\Rightarrow F$ is a primitive of f on $[-1,1]$

But f is not integrable on $[-1,1]$ because f is not bounded on $[a,b]$

Change of Variable in a Riemann Integral

The formula $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$ previously proved assumes the form

$$\int_a^b f(x)\,dx = \int_A^B f[g(t)]\,g'(t)\,dt$$

When $\alpha(x) = x$ & g is a strictly monotone function with a continuous derivative. It is valid if $\in R[a,b]$. When f is continuous we remove the restriction that g is monotone.

<u>Theorem#</u> (Change of Variable)

Suppose that U has continuous derivative on $[c,d]$. Let f be continuous on the range of U. Let $U(c)=a$, $U(d)=b$. Then

$$\int_a^b f(x)\,dx = \int_c^d f[U(t)]\,U'(t)\,dt$$

<u>OR</u>

Let U be continuously differentiable function on $I=[c,d]$. Let $U(c)=a$, $U(d)=b$. Let f be continuous on the $U(I)$ i.e range of U, Then

$$\int_a^b f(x)\,dx = \int_c^d f[U(t)]\,U'(t)\,dt$$

<u>Proof</u> $\because$ The range of a continuous function on a closed and bounded is also a closed and bounded interval

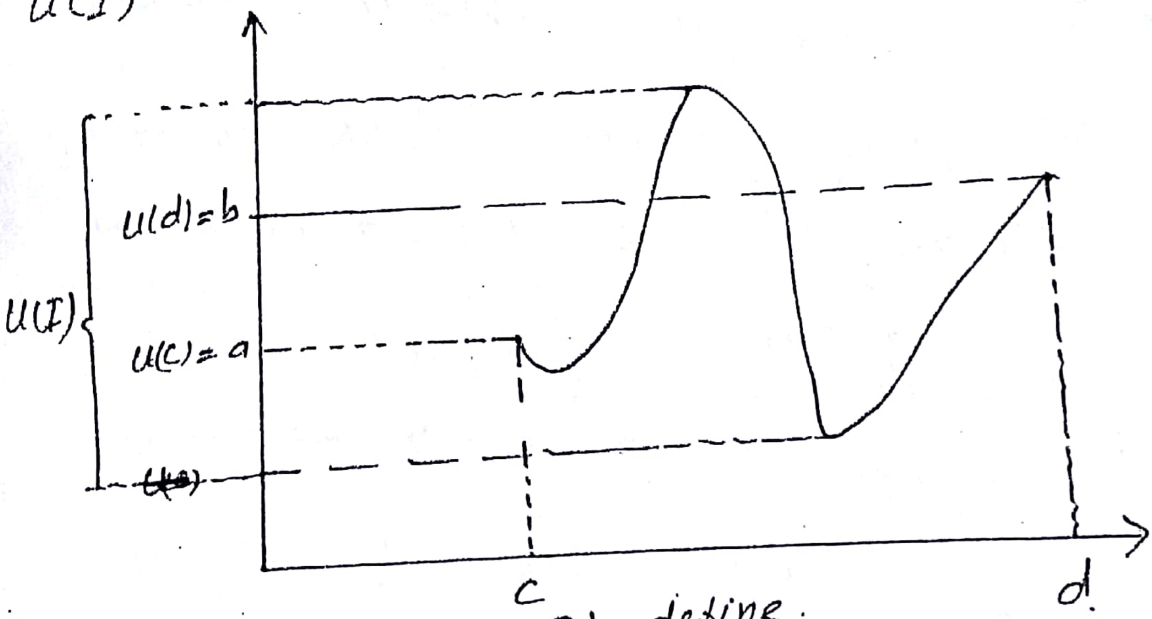
$\therefore$ U has a range which is also a closed and interval

Since composite function of two continuous function is also continuous, therefore $f(U)$ is continuous on $[c,d]$ and hence.

integrable because Continuous function is integrable.

Again $(f \circ u)u' = f(u)u'$ being a product of two continuous functions is continuous on $[c, d]$

Note that u is not assumed to be monotone so that $u(I)$ contains, but may not equal, the interval with end points $u(c) = a$ & $u(d) = b$. However since f is assumed continuous on the interval $u(I)$, it is integrable on every sub-interval of $u(I)$. See fig



For all $x \in u(I)$, define.

$$F(x) = \int_a^x f(t)\,dt \longrightarrow ①$$

Also define G on $[c, d]$ as follows

$$G(x) = \int_c^x f[u(t)]\,u'(t)\,dt \longrightarrow ②$$

We show that

$$F(u(x)) = G(x)$$

$\because f(u)u'$ is continuous

$\therefore$ By property of integral function. derivative of G is equal to $f(u)u'$ on $[c, d]$ i.e

$$G'(x) = f[u(x)]u'(x) \quad \forall x \in [c,d] \rightarrow (3)$$

But by chain rule.

$$(F \circ u)'(x) = F'[u(x)]u'(x) = f[u(x)]u'(x) \quad \forall x \in [c,d] \rightarrow (4)$$

By (3) & (4)

$$G'(x) = F'[u(x)]$$

$\because F'(x) = f(x) \quad \therefore G'(x) = f(x)$

$\therefore G(x)$ & $F[u(x)]$ are two primitives of f on $[c, d]$

But any two primitives differ by a constant D such that

$$F(u(x)) = G(x) + D \longrightarrow (5)$$

$$\Rightarrow F[u(x)] = \int_a^{u(x)} f(t)dt = \int_c^x f[u(t)]u'(t)dt + D$$

Let $x = c$ so that $u(c) = a$, we obtain.

$$\int_a^a f(t)dt = \int_c^c f[u(t)]u'(t) + D$$

$$0 = 0 + D \Rightarrow D = 0$$

By (5)

$$F[u(x)] = G(x) \quad \forall x \in [c, d]$$

So $\quad F[u(d)] = G(d)$

$$\Rightarrow F(b) = G(d)$$

$$\Rightarrow \int_a^b f(t)dt = \int_c^d f[u(t)]u'(t)dt$$

OR

$$\int_a^b f(x)dx = \int_c^d f[u(t)]u'(t)dt$$

Integration by Parts#

Theorem# If F and G are primitives of f and g respectively and f, g are Riemann-integrable functions, Then

$$\int_a^b F(x)\, g(x)\, dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)\, G(x)\, dx$$

Proof# $\because$ Primitive of a function is continuous

$\therefore$ F, G are continuous on $[a, b]$

Again a continuous function on $[a, b]$ is Riemann integrable on $[a, b]$, therefore F, G are Riemann integrable on $[a, b]$.

$\because$ Product of two Riemann integrable on $[a, b]$ is Riemann integrable

$\therefore$ Fg & fG are integrable on $[a, b]$

$\Rightarrow$ The two integral in the theorem exist

Let $H(x) = F(x)\, G(x)$

$$H'(x) = F'(x)\, G(x) + F(x)\, G'(x)$$

$$= f(x)\, G(x) + F(x)\, g(x) = h(x) \text{ say}$$

$\because$ F, G are integrable

$\therefore$ $H = FG$ is also integrable.

Also fG, Fg are integrable. and $h = fG + Fg$ is integrable on $[a, b]$

$\because$ $H'(x) = h(x)$

$\therefore$ $H(x)$ is a primitive of $h(x)$

By fundamental theorem of Calculus

$$\int_a^b h(x)\, dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b [f(x)G(x) + F(x)g(x)]dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b f(x)G(x)dx + \int_a^b F(x)g(x)dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b F(x)g(x)dx = H(b) - H(a) - \int_a^b f(x)G(x)dx$$

$$= F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Question # Show that $f(x) = \sin x$ is Riemann integrable over $[0, \pi/2]$

Sol Take $P = \{0, \frac{\pi}{2n}, \frac{\pi}{n}, \frac{3\pi}{2n}, \ldots \frac{n\pi}{2n}\}$

by dividing $[0, \frac{\pi}{2}]$ into n equal parts

$\sup f = M_k$ on $[x_{k-1}, x_k]$ $\qquad \Delta x = \frac{\pi}{2n}$

$\inf f = m_k$ on $[x_{k-1}, x_k]$

Then $M_k = \sin \frac{k\pi}{2n}$ $\qquad m_k = \sin \frac{(k-1)\pi}{2n}$

$$U(P,f) - L(P,f) = \sum_{k=1}^{n} \left(\sin\frac{k\pi}{2n} - \sin\left(\frac{k-1}{2n}\pi\right)\right)\cdot\frac{\pi}{2n}$$

$$\leq \frac{\pi}{2n} < \epsilon \qquad \text{for } n > \frac{\pi}{2\epsilon} = n_0$$

$\Rightarrow f$ is Riemann integrable over $[0, \frac{\pi}{2}]$

By Muhammad Hussain.
Lecturer Maths
Govt. College Asghar Mall
Rawalpindi

Mean Value Theorems of Integral

Theorem # If f is continuous on $[a, b]$ and α is increasing on $[a, b]$, then there exists a point c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)\int_a^b d\alpha = f(c)\left(\alpha(b)-\alpha(a)\right)$$

Proof # If $\alpha(a) = \alpha(b)$, then theorem is trivially true because both sides are zero and any value of c in $[a, b]$ gives the result.

If f is constant on $[a, b]$ say $f(x) = k$ $\forall x \in [a, b]$, then $\int_a^b f\,d\alpha = k[\alpha(b)-\alpha(a)]$

Hence for any $c \in [a, b]$, we have.

$$\int_a^b f\,d\alpha = [\alpha(b)-\alpha(a)]\,f(c)$$

We assume that $\alpha(a) < \alpha(b)$

$\because$ f is continuous on $[a, b]$

$\therefore$ f attains its sup & inf on $[a, b]$ & $f \in R(\alpha)$

Let $M = \sup f$ on $[a, b]$

$m = \inf f$ on $[a, b]$

Then

$$m[\alpha(b)-\alpha(a)] \le \int_a^b f\,d\alpha \le M[\alpha(b)-\alpha(a)]$$

$$\Rightarrow m \le \frac{\int_a^b f\,d\alpha}{\alpha(b)-\alpha(a)} \le M$$

If $\dfrac{\int_a^b f\,d\alpha}{\alpha(b)-\alpha(a)}$ is equal to either of M, m, then

theorem is proved because M, m are values of f at some point of $[a,b]$ and that point will be point c required by Theorem.

If $\dfrac{\int_a^b f\,d\alpha}{\alpha(b)-\alpha(a)}$ is not equal to M, m, then

$$m < \frac{\int_a^b f\,d\alpha}{\alpha(\alpha)-\alpha(a)} < M.$$

By extreme-value theorem there are points $x_1, x_2 \in [a\ b]$ such that

$$f(x_1) = m \qquad f(x_2) = M$$

Thus
$$f(x_1) < \frac{1}{\alpha(b)-\alpha(a)} \int_a^b f\,d\alpha < f(x_2)$$

By intermediate value theorem $\exists$ a point $c \in (x_1, x_2) \subseteq [a,b]$ such that

$$\frac{\int_a^b f\,d\alpha}{\alpha(b)-\alpha(a)} = f(c)$$

$$\Rightarrow \int_a^b f\,d\alpha = f(c)\,\{\alpha(b)-\alpha(a)\}$$

Theorem # (1st Mean Value Theorem)

If f is continuous on $[a,b]$, $g \in R_\alpha[a,b]$ and g keeps same sign on $[a,b]$, α is increasing on $[a,b]$, then $\exists$ a point $c \in [a,b]$ such that

$$\int_a^b fg\,d\alpha = f(c)\int_a^b g\,d\alpha.$$

Proof # $\because$ f is continuous on $[a.b]$

$\therefore$ $f \in R_\alpha[a,b]$

Also $g \in R_\alpha[a\ b]$

$\Rightarrow fg \in R_\alpha[a,b]$

Let $g(x) \geq 0 \quad \forall x \in [a,b]$

$\because f$ is continuous on $[a,b]$

$\therefore f$ attains its sup & inf on $[a,b]$

Let $M = \sup f = f(x_2) \qquad x_2 \in [a,b]$

$\quad m = \inf f = f(x_1) \qquad x_1 \in [a,b]$

Then
$$m \leq f(x) \leq M \qquad \forall x \in [a,b]$$

$\because g(x) \geq 0$

$$\Rightarrow mg(x) \leq f(x)g(x) \leq Mg(x) \quad \forall x \in [a,b]$$

$\Rightarrow$ Integrating
$$m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx$$

$$\Rightarrow m \leq \frac{\int_a^b fg\,d\alpha}{\int_a^b g(x)\,d\alpha} \leq M$$

If $\dfrac{\int_a^b fg\,d\alpha}{\int_a^b g(x)\,d\alpha} = M$ or m, then proved. otherwise by intermediate value theorem $\exists$ a point $c \in (x_1, x_2) \subseteq [a,b]$ such that

$$\frac{\int_a^b fg\,d\alpha}{\int_a^b g(x)\,d\alpha} = f(c)$$

$$\Rightarrow \int_a^b fg\,d\alpha = f(c)\int_a^b g(x)\,d\alpha$$

<u>Note</u> Theorem also holds if $g(x) \leq 0 \quad \forall x \in [a,b]$

If we take $g(x) = 1$, then $g \in R_\alpha[a,b]$ $g > 0$

& we have.
$$\int_a^b f\,d\alpha = f(c)[b-a]$$

Theorem (For Riemann integral)

If f is continuous on $[a,b]$, then $\exists$ a point c in $[a,b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

Proof # If $a=b$ or $f=$ constant, then theorem is trivially proved. Let $a\neq b$, $f\neq$ constant.

$\because$ f is continuous

$\therefore$ $f\in R[a,b]$ & attains its sup and inf on $[a,b]$.

Let $x_1, x_2 \in [a,b]$ such that

$f(x_1) = m = \inf f \quad f(x_2) = M = \sup f$

Now

$$m(b-a) \le \int_a^b f\,dx \le M(b-a)$$

$$\Rightarrow m(b-a) \le \int_a^b f\,dx \le M(b-a)$$

$$\Rightarrow \quad m \le \frac{\int_a^b f\,dx}{b-a} \le M$$

If $\dfrac{\int_a^b f\,dx}{b-a} = M$ or m, then theorem is proved.

otherwise by intermediate value Theorem $\exists$ a point $c \in (x_1, x_2) \subseteq [a,b]$ such that

$$\frac{\int_a^b f\,dx}{b-a} = f(c)$$

$$\Rightarrow \int_a^b f\,dx = f(c)\,(b-a)$$

Note $\dfrac{1}{b-a}\int_a^b f\,dx$ is called mean value of the function on $[a,b]$. This theorem says that a continuous function always assumes its mean value.

2nd Mean Value Theorem OR Bonnet's Mean Value Theorem

Assume That α is Continuous and $\alpha \uparrow$ on $[a, b]$, f is monotonically increasing on $[a, b]$, Then There exists a point $x_0 \in [a, b]$ Such That

$$\int_a^b f(x)\, d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$$

Proof # $\because$ α is Continuous on $[a, b]$

$\therefore$ α is in $R_f[a, b]$

Also $f \in R_\alpha[a, b]$ and by parts integration we have

$$\int_a^b f(x)\, d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\, df(x) \longrightarrow (1)$$

Now in integral $\int_a^b \alpha(x)\, df$, α is Continuous and f is $\uparrow$ on $[a, b]$. Therefore by above mean value Theorem there exists a point $x_0 \in [a, b]$ such That

$$\int_a^b \alpha(x)\, df = \alpha(x_0)[f(b) - f(a)]$$

using in (1)

$$\int_a^b f\, d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \alpha(x_0) f(b) + \alpha(x_0) f(a)$$

$$= f(a)[\alpha(x_0) - \alpha(a)] + f(b)[\alpha(b) - \alpha(x_0)]$$

$$= f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$$

Corollary# If f is continuous and monotone increasing on $[a,b]$, there exists a point c in $[a,b]$, such that

$$\int_a^b f(t)\,dt = f(a)(c-a) + f(b)(b-c)$$

Proof# In above theorem take $f(t)$ and $\alpha(t) = t$ in place of α, then for $x_0 = c$

$$\int_a^b f(t)\,dt = f(a)(c-a) + f(b)(b-c)$$

Theorem# (2nd MVT for Riemann Integration)
If f is continuous and increasing on $[a,b]$ and g is non-negative and integrable on $[a,b]$, then $\exists$ a point $x_0 \in [a,b]$ such that

$$\int_a^b fg\,dx = f(a)\int_a^{x_0} g\,dx + f(b)\int_{x_0}^b g\,dx.$$

Proof# By 1st Mean value theorem $\exists$ a point $y \in [a,b]$ such that

$$\int_a^b fg\,dx = f(y)\int_a^b g\,dx \longrightarrow (1)$$

Also since f is increasing, therefore

$$f(a) \le f(y) \le f(b)$$

Consider the function G defined on $[a,b]$ by

$$G(x) = [f(b) - f(a)]\int_x^b g\,dx$$

$\because$ g is non-negative, G is a decreasing function.

Further

$$G(a) = [f(b) - f(a)]\int_a^b g\,dx$$

$$\ge [f(y) - f(a)]\int_a^b g\,dx$$

$$\ge 0 = G(b) \qquad \Rightarrow G(a) \ge G(b)$$

$\because$ G is continuous on $[a, b]$

$\therefore$ By intermediate value theorem $\exists$ a point $x_0 \in [a, b]$ such that

$$G(x_0) = [f(y) - f(a)] \int_a^b g\,dx$$

Now

$$G(x_0) = [f(b) - f(a)] \int_{x_0}^b g\,dx = [f(y) - f(a)] \int_a^b g\,dx$$

we have

$$f(b) \int_{x_0}^b g\,dx - f(a) \int_{x_0}^b g\,dx = f(y) \int_a^b g\,dx - f(a) \int_a^b g\,dx$$

$$= \int_a^b f(x)\, g(x)\,dx - f(a) \int_a^b g\,dx$$

$$\Rightarrow \int_a^b fg\,dx = f(a) \left[\int_a^b g\,dx - \int_{x_0}^b g\,dx \right] + f(b) \int_{x_0}^b g\,dx$$

$$= f(a) \int_a^{x_0} g\,dx + f(b) \int_{x_0}^b g\,dx$$

Question # prove that $\frac{\pi^2}{9} \le \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \le \frac{2\pi^2}{9}$

Sol let $f(x) = \frac{1}{\sin x}$ & $g(x) = x$

Then f, g are continuous on $[\pi/6, \pi/2]$ & hence integrable on $[\pi/6, \pi/2]$

Also $g(x) = x > 0 \quad \forall x \in [\pi/6, \pi/2]$

Since f decreasing on $[\pi/6, \pi/2]$

$\inf f = f(\pi/2) = 1 \qquad \sup f = f(\pi/6) = 2$

By 1st Mean value theorem $\exists$ a point $c \in [\pi/6, \pi/2]$ such that

$$\int_{\pi/6}^{\pi/2} f(x)\, g(x)\,dx = f(c) \int_{\pi/6}^{\pi/2} g(x)\,dx$$

$$\Rightarrow \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx = f(c) \int_{\pi/6}^{\pi/2} x\,dx = f(c) \cdot \frac{\pi^2}{9}$$

$$\Rightarrow \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx = f(c)\cdot\frac{\pi^2}{9}$$

$\because$ f is continuous & decreasing on $[\pi/6, \pi/2]$ & $c \in [\pi/6, \pi/2]$

$\therefore$ $\pi/6 \le c \le \pi/2$

$\Rightarrow f(\pi/6) \geq f(c) \geq f(\pi/2)$

$\Rightarrow 1 \le f(c) \le 2$

$$\Rightarrow 1 \le \frac{9}{\pi^2}\int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \le 2$$

$$\Rightarrow \frac{\pi^2}{9} \le \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \le \frac{2\pi^2}{9}$$

By Muhammad Hussain
Assistant Professor (Maths)
Govt. College Asghar Mall.
Rawalpindi